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Completeness and stability of spacetimes with Lorentz-Minkowski ends

Completitud y estabilidad de espaciotiempos con finales Lorentz-Minkowski

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Declaro explícitamente que el trabajo presentado es original, entendido en el sentido de que no he utilizado fuentes sin citarlas debidamente.

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Resumen

En este trabajo se estudia la completitud geodésica y su estabilidad en variedades lorentzianas. Comenzamos contextualizando el trabajo e indicando las nociones básicas de variedades diferenciables que se van a utilizar. Posteriormente se estudiará la holonomía de una variedad afín, así como un resultado sobre completitud geodésica en este tipo de variedades. Después, se define una topología en el espacio de las métricas lorentzianas de una variedad, lo que da una base para el estudio de la estabilidad de la completitud y de la incompletitud. Se presentan condiciones suficientes para la estabilidad de estas dos propiedades. Finalmente, recopilando lo que se ha estudiado en los anteriores capítulos, nos planteamos si existen variedades lorentzianas en las que podamos omitir alguna de las condiciones suficientes de estabilidad o si, de lo contrario, podemos relajar los criterios de estabilidad. Éste será el caso de espaciotiempos con varios finales tipo Lorentz-Minkowski, esto es, variedades lorentzianas que, fuera de un compacto, tienen componentes isométricas al espacio de Lorentz-Minkowski. Para encontrar resultados de estabilidad, las conexiones próximas a la de Levi-Civita y las de la segunda forma fundamental de la región compacta serán esenciales. Sobre la región compacta, se introducirá un concepto de completitud que se verificará automáticamente si su holonomía es precompacta. Finalmente, se obtendrá un resultado general de completitud y estabilidad para espaciotiempos con varios finales Lorentz-Minkowski.

Palabras clave: Geometría Lorentziana, Holonomía precompacta, Completitud geodésica, Estabilidad de la completitud, Final Lorentz-Minkowski, Topologías finas $\mathscr{C}^r$.

Abstract

In this work, the properties of completeness and stability of completeness in Lorentzian manifolds are studied. We start contextualizing the work and introducing basic notions of differentiable manifolds. Then, holonomy of affine manifolds will be studied, as well as a result on geodesic completeness in this kind of manifolds. After this, we will define a topology in the space of Lorentzian metrics, in order to provide a basis to study stability of completeness and incompleteness. Sufficient conditions for the stability of these properties are presented. Finally, using the results studied in the previous chapters, we wonder if in some kind of Lorentzian manifolds we can omit one of the stability conditions, or if we can relax the stability criteria. In this last case, we find spacetimes with several Lorentz-Minkowski ends, that is, Lorentzian manifolds that, out of a compact set, have isometric components to Lorentz-Minkiowski space. To find stability results in this kind of manifolds, the Levi-Civita connection and the second fundamental form become essential. In the compact region, a completeness concept will be introduced, that will be directly verified if its holonomy is precompact. In the end, a general result on completeness and stability will be obtained for spacetimes with several Lorentz-Minkowski ends.

Key words: Lorentzian Geometry, Precompact holonomy, Geodesic completeness, Stability of completeness, Lorentz-Minkowski end, Fine $\mathscr{C}^r$ topologies.

Contents

Introduction

Geodesic completeness is an essential property of Riemannian and semi-Riemannian manifolds, that has been widely studied. We need to have in mind that the concepts of metric completeness and geodesic completeness are clearly different. In Riemannian geometry, both are equivalent, whereas in semi-Riemannian we have to make a distinction. The target of study along this work will be geodesic completeness of Lorentzian manifolds. This property serves as a tool to study different aspects of a manifold. For example, geodesic completeness is a key tool to classify Lorentzian spaceforms and it lies in the core of relativistic singularity theorems , where curvature criteria and completeness interplay to obtain focal points and consequences on the global structure of the manifold, as in the celebrated theorems by Hawking and Penrose (see [3] for a review).

As stated above, geodesic and metric completeness are equivalent in Riemannian geometry. The Hopf-Rinow theorem not only provides this equivalence, but it also ensures that compactness implies completeness. In the semi-Riemannian case, this theorem does not hold and there is not analogous theorem to it. In fact, it is remarkable that compact manifolds can fail to be geodesically complete. As in semi-Riemannian geometry no well defined distance is induced by a metric, the study of geodesic completeness rather than metric completeness becomes primordial. For example, geodesic completeness becomes an essential property to classify compact Lorentzian spaceforms. So, the proof of its completeness by Carriére [4] in the flat case, extended to constant curvature by Klinger [10], was a milestone in the classification problem. It is still open the proof of completeness in the case conformal to constant curvature. However, the completeness of compact Lorentzian manifods with Abelian geometry has been proved recently in [11].

Geodesic completeness is a global property that can be altered under a small change. Being a global property, geodesic completeness is lost in any semi-Riemannian complete manifold if a point is removed. A question along this work is if a small change in the metric can also alter the geodesic completeness or incompleteness. This cannot occur in the Riemannian case but, as we will see, sometimes a small change in a complete Lorentzian metric make an incomplete Lorentzian metric, and conversely. Nevertheless, under some conditions, we can ensure that both complete and incomplete metrics have arbitrarily close complete, or respectively, incomplete metrics.

As geodesic completeness depends only on the Levi-Civita connection, along this work, we will consider also properties of non-necessarily metrizable affine connections. Indeed, the holonomy group of a connected affine manifold and the second fundamental form will be two central ingredients for the study of completeness and stability of completeness.

This work is addressed to give the proof of the following theorem that will be proven at the end of the document.

Theorem. Let (M,g) be a Lorentzian manifold with m Lorentz-Minkowski ends. If $M\setminus\mathring{K} = \cup_{i=1}^{m}(\mathbb{L}^n\setminus\mathring{K}_i)$ and K has precompact holonomy, there exists a compact set $\bar{K}$ that contains K, such that there exists $U_{\bar{K}}(g)$ a neighborhood of metrics up to $\bar{K}$ such that $(M,\bar{g})$ is complete for $\bar{g}\in U_{\bar{K}}(g)$.

This work is structured in four chapters. The first chapter is called *Preliminaries*. In it, general concepts of differentiable manifolds are included. Lorentzian geometry is defined, as well as some definitions and characterizations of Lorentzian manifolds. In the second one, *Precompact holonomy and completeness*, the primordial objective is to prove a theorem by Aké and Sánchez in [5]; if a manifold has precompact holonomy and it is compact, then it is complete. To achieve this result, it is necessary to study properties of the holonomy groups. In chapter 3, *Stability of completeness and incompleteness*, we define the concept of stability and the fine $\mathscr{C}^r$ topologies in the space of metrics. Then, we state some results and examples on stability of incompleteness and completeness. The results in stability were developed by Beem and Ehrlich in [1].

The main purpose of this work is developed in the last chapter, *Completeness in manifolds close to* $\mathbb{L}^n$, and it consists on studying completeness of some characteristic Lorentzian manifolds when the hypotheses of the theorems of stability of completeness fail. In this chapter, results of increasing complexity are presented, in order to achieve a more general result. First, we consider manifolds with a Lorentz-Minkowski end, that is, isometric to $\mathbb{L}^n$ out of a compact set. Convexity arguments will be used in order to ensure completeness. The first case studied is Lorentzian manifolds that have a spherical Lorentz-Minkowski end. As the conditions of the general stability are not fulfilled, we will introduce a new kind of neighborhoods, therefore, we will relax the stability criteria by fixing the metric in the compact complete set. In the case that the compact set is a general compact connected set, we will need the additional condition of precompact holonomy in order to ensure completeness and stability of completeness. Finally, we will define manifolds with several Lorentz-Minkowski ends, that is, manifolds that out of a compact set are isometric to some copies of $\mathbb{L}^n$ and we will give a result on completeness and stability in this case, to finally arrive to the theorem stated above.

Chapter 1

Preliminaries

Ancient geometry was closely related to the study of shapes inside the space. That is, there was no point in studying spaces with dimension greater than 3. However, having generalized the Euclidean space to an arbitrary dimension, $\mathbb{R}^n$, there was a possibility to extend geometry. A way to generalize curves and surfaces consists on the definition of manifolds, as topological spaces locally homeomorphic to $\mathbb{R}^n$. That is, a manifold is a topological space endowed with an atlas, which consists of homeomorphisms to open sets of $\mathbb{R}^n$. Having the concept of manifold in hand, a whole theory has been developed. In this chapter, we introduce some of the definitions and relevant results in manifolds that will be essential for the subsequent analysis. Secondly, we will introduce semi-Riemannian metrics so as to give results that will be used in the following chapters. The contents of this chapter have been studied in the Master and the Degree of Mathematics, and are included in this work to fix the concepts and introduce notation.

1.1 Differentiable manifolds

As aforementioned, manifolds have been the backbone of the development of Geometry over the last two centuries. Along this section, some relevant definitions and results are introduced, others, that will be excluded, can be found in [6].

Definition 1.1. *A* differentiable manifold *is a topological set provided with an atlas, that is, a set of homeomorphisms* $\mathscr{A} = \{(U_i, \mathbf{x}_i)\mathbf{x} : U_i \subseteq M \to \mathbf{x}(U_i) \subseteq \mathbb{R}^n\}$ *whose domains cover the manifold and fulfill the following condition. Given two maps* $\mathbf{x} : U \subset M \to \mathbb{R}^n$ *and* $\mathbf{y} : V \subset M \to \mathbb{R}^n$ *with* $U \cap V \neq \emptyset$*, the map* $y \circ x^{-1} : \mathbf{x}(U \cap V) \to \mathbf{y}(U \cap V)$ *is a diffeomorphism in* $\mathbb{R}^n$.

When we talk about differentiable manifolds, we will assume that are Hausdorff and that have countable bases. In this chapter, we will assume that the manifolds are connected. In the study of differentiable manifolds differentiability is linked to $\mathscr{C}^\infty$ differentiability. However along this work, we will require $\mathscr{C}^2$ differentiability. The differentiable manifolds with boundary are defined as the differentiable manifolds but the open sets are taken in $\mathbb{R}^n_+ = [0, \infty) \times \mathbb{R}^{n-1}$. The tangent space and vector fields are a key tool along this work. Some concepts must be introduced before. In the next definition differentiable applications are introduced.

Definition 1.2. *Let* $f : M \to \mathbb{R}$ *be a function. We say that* f *is a* differentiable function *if for* *every* $p \in M$ *with* $p \in U$ *a coordinate chart* $(U, \mathbf{x})$*, the function* $f \circ \mathbf{x}^{-1} : U \to \mathbb{R}$ *is differentiable in* $\mathbb{R}^n$*. The set of differentiable functions is* $\mathscr{F}(M)$*.*

If $g : M \to N$ *is an application, we say that it is a* differentiable application *if for every point* p *there exist coordinate charts* $(U_p, \mathbf{x})$ *in* M *and* $(V_{g(p)}, \mathbf{y})$ *in* N *such that* $\mathbf{y} \circ g \circ \mathbf{x}^{-1} : U_p \to V_{g(p)}$ *is a differentiable application from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$*.*

If M, N *are differentiable manifolds of dimension* n*,* $h : M \to N$ *is bijective and* h, h^{-1} *are differentiable, we say that* h *is a* diffeomorphism.

We say that $\gamma : I \to M$*, where* I *is an interval, is a* differentiable curve *if for every* $t \in I$ *there is a coordinate chart* $(U, \mathbf{x})$ *such that* $\mathbf{x} \circ \gamma$ *is differentiable in* $\mathbb{R}$*. We say that a curve is* piecewise differentiable *if* I *is compact and*γ *is differentiable in* $I \setminus \{t_1, t_2, \ldots, t_m\}$ *and the partial derivatives exists at each* t_i*,* $i = 1, \ldots, m$*.*

We define the partial derivative of a function $f : M \to \mathbb{R}$ in a point in a coordinate chart $\mathbf{x}$ as: $\frac{\partial f}{\partial x^i} = \frac{\partial (f \circ \mathbf{x}^{-1})}{\partial u^i}$, where $u^1, \ldots, u^n$ are the coordinates of $\mathbf{x}(U)$ in $\mathbb{R}^n$. This definition is purely analytical. In every differential manifold of dimension greater than 0, curves can be written as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. The derivative of a function in a point $p = \gamma(0)$ along this curve is given by the following expression:

$$\gamma'(0) f = \frac{d(f \circ \gamma)}{dt} = \frac{d(f \circ \mathbf{x}^{-1}) \circ (\mathbf{x} \circ \gamma)}{dt} = \sum_i (\gamma^i)' \frac{\partial f}{\partial x^i} \tag{1.1}$$

This is the key to regard tangent vectors as either as directional derivatives at each point, or classes of equivalence of curves, which will be assumed in the remainder.

Tangent space and vector fields

Having the concept of derivative along a curve and the tangent vector, the tangent space of M at $p \in M$ is defined by

$$T_p M = \{\gamma'(0) : \gamma \text{ is a curve in M with } \gamma(0) = p\}, \tag{1.2}$$

which is endowed with the natural structure of real vector $n-$space obtained when looking these vectors as directional derivatives. Then, given a coordinate chart in p, a basis of $T_p M$ is $\left\{ \frac{\partial}{\partial x^1}|_p, \ldots, \frac{\partial}{\partial x^n}|_p \right\}$. In fact, this is the basis of $T_p M$ for every p in the same coordinate chart, and thus, we will omit the subindex p. And then, the tangent bundle of a manifold is the union of the tangent spaces for every point in the manifold.

$$TM = \cup_{p \in M} T_p M$$

In fact, the tangent bundle has a structure of differentiable manifold itself with dimension $2n$ (see [6]). To complete this definition, a canonic projection must be added, as:

$$\begin{array}{rccc} \pi : & TM & \to & M \\ & v & \mapsto & p \end{array}$$

which gives a new definition of the tangent space of a point as a fiber of the tangent bundle, that is $T_p M = \pi^{-1}(p)$. Therefore, we are ready to define the vector fields in a manifold as sections of the tangent bundle:

$$\begin{array}{rccc} X : & M & \to & TM \\ & p & \mapsto & v \end{array},$$

which means that $\pi \circ X = id_M$. To say that a vector field is differentiable, we add the condition that X is differentiable. The set of vector fields in M is denoted as $\mathfrak{X}(M)$ and has a natural structure of a module on the $\mathscr{F}(M)$ ring of the smooth functions on M.

Cotangent space and one-forms

The dual of a vector space V is usually denoted as V^*. As T_pM is a vector space, we can define the dual of T_pM. In each point, the cotangent space is defined as

$$T_p^*M = (T_pM)^*, \tag{1.3}$$

The elements of T_p^*M are covectors and are defined as linear functions $\omega_p : T_pM \to \mathbb{R}$. The cotangent bundle T^*M is defined as

$$T^*M = \cup_{p\in M} T_p^*M,$$

which also becomes a vector bundle of the same dimension as TM. The sections of this bundle are called one-forms, which are dual to vector fields. A special kind of one-forms are the differential of functions. The differential of a function $f \in \mathscr{F}(M)$ is a one-form df such that $df(v) = v(f)$, where $v(f)$ is the derivation of f in the direction of v, for every $p \in M$ and $v \in T_pM$. A basis of T_p^*M can be given by a basis $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ of T_pM with p in a coordinate chart as its dual $\{dx^1, \dots, dx^n\}$.
Usually, the set of one-forms is denoted as $\Lambda^1(M)$, but for the definition of tensors, it is more illustrating to call them $\mathfrak{X}^*(M)$.

Tensor fields

Tensor field theory is based on algebra of tensors and the definition of vector fields and one forms.

Definition 1.3. *A tensor field of type (r,s) is a $\mathscr{F}(M)$-multilinear function:*

$$\begin{array}{rccc} T: & \mathfrak{X}^*(M)\times \overset{r}{\ldots} \times\mathfrak{X}^*(M)\times\mathfrak{X}(M)\times \overset{s}{\ldots} \times\mathfrak{X}(M) & \to & \mathscr{F}(M) \\ & (\omega^1,\dots\omega^r, X_1,\dots,X_s) & \mapsto & T(\omega^1,\dots\omega^r, X_1,\dots,X_s). \end{array}$$

A tensor of kind $(0,0)$ is a function $f \in \mathscr{F}(M)$.

Note that tensor fields can be written in the usual tensorial notation as:

$$Y_1 \otimes \ldots \otimes Y_r \otimes \theta_s \otimes \ldots \theta^s(\omega^1,\dots\omega^s, X_1,\dots,X_s) = \omega^1(Y_1)\cdot\ldots\cdot\omega^r(Y_r)\cdot\theta^1(X_1)\cdot\ldots\cdot\omega^r(X_r).$$

For two differentiable manifolds M, N an application $f : M \to N$ induces two applications:

- $f_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$, such that for any $p \in M$ and for every function $g \in \mathscr{F}(N)$ $f_*(v)(g) = v(g \circ f)$.
- $f^* : \mathfrak{X}^*(N) \to \mathfrak{X}^*(M)$, such that for $p \in N$ with $q = f(p)$, for any vector v in T_pM, $(f^*\omega)(v) = \omega(f_*(v))$. A form $f^*(\omega)$ is called the pullback of ω.

This notions can be extended to tensor fields using tensorial products.

1.1.1 Metrics

Metrics in differentiable manifolds work as a generalization of scalar products. Take for example, $\mathbb{R}^n$ with the global chart that maps $\mathbb{R}^n$ to itself and the usual scalar product in each point. In this case, we would have a manifold provided with a metric given by the usual scalar product. However, metrics do not need to be positive definite as scalar products are. The formal definition of a metric follows.

Definition 1.4. *Let M be a differential manifold. A metric g in M is a symmetric nondegenerate tensor field of kind* $(0,2)$.

For each point, this metric is written as $g_p : T_pM \times T_pM \to \mathbb{R}$. The condition of nondegeneration is the same as the nondegeration of bilinear forms. This means that if a vector u fulfills $g(u,v) = 0$ for all $v \in T_pM$, then $u = 0$. This scalar product does not have to be positive definite but it needs to have the same signature in each point, because of the nondegenerate condition. Hence, metrics are classified by their signature. Let (M,g) be a manifold of dimension n provided with a metric. As g is a bilinear form, for any coordinate system $g = \sum_{ij} g_{ij} dx^i \otimes dx^j$. In fact, in every point a basis of vectors can be found such that $g_{ij} = \pm\delta_{ij}$. Let p, q be the indices of the metric, such that p is the number of vectors v_i of the basis that $g_{ii} = 1$ and q is the number of vectors that $g_{ii} = -1$.

- If $q = 0$, g is a Riemannian metric and (M,g) a Riemannian manifold. In this special case, g is positive definite for every $p \in M$. In [6], it is proven that every differentiable manifold admits a Riemannian metric.
- If $q = n$, g is negative definite, and thus, $(M,-g)$ is a Riemannian manifold.
- If $1 \leq p, q \leq n-1$, g is a semi-Riemannian metric and (M,g) is a semi-Riemannian manifold.

In the particular case that $q = 1$ and $p \geq 1$, we say that g is a Lorentzian metric, and thus, (M,g) a Lorentzian manifold. A typical example that will be widely studied for this type of manifolds consists on $\mathbb{L}^n$. It is defined as $\mathbb{R}^n$ provided with the following metric:

$$\eta = -dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + \ldots + dx^n \otimes dx^n. \tag{1.4}$$

Two manifolds can be related in terms of its metric.

Definition 1.5. *Let (M,g) and $(N,\bar{g})$ be two differentiable manifolds having the same dimension. If $f : M \to N$ is a differentiable application such that for $p \in M$ and every $v, w \in T_pM$, $g(v,w) = \bar{g}(df_p(v), df_p(w))$, we say that f is a* local isometry.
If, in addition, f is a diffeomorphism, we say that f is an isometry.

If two manifolds are isometric, they have the same metric tensor g, that is, $g = f^*\bar{g}$, the metric of M is the pullback metric of N. Another important kind of applications are conformal transformations.

Definition 1.6. *Let Let (M,g) and $(N,\bar{g})$ be two differential manifolds having the same dimension. If $f : M \to N$ is a differentiable application such that for $p \in M$ and every $v, w \in T_pM$, $\Omega g(v,w) = \bar{g}(df_p(v), df_p(w))$ for $\Omega : M \to (0,\infty)$, we say that f is a* conformal transformation. *In case $\Omega \in \mathbb{R}$, that is, if Ω is a constant, f is a* homothety.

Notice that if $\Omega \equiv 1$, the conformal transformation is an isometry.

1.1.2 Affine connections

The affine connections are introduced as a tool to differentiate vector fields in the direction of others.

Definition 1.7. *Let M be a differentiable manifold and $X, Y \in \mathfrak{X}(M)$. An affine connection of a X respect Y is an application:*

$$\begin{array}{rccc} \nabla: & \mathfrak{X}(M) \times \mathfrak{X}(M) & \to & \mathfrak{X}(M) \\ & (X, Y) & \mapsto & \nabla_X Y \end{array}$$

that has the following properties:

1. $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$.
2. $\nabla_X (Y_1 + Y_2) = \nabla_X Y + \nabla_X Y_2$.
3. $\nabla_{fX} Y = f \nabla_X Y$.
4. $\nabla_X f Y = f \nabla_X Y + X(f) Y$

In light of the property 4, we can see that ∇ is not $\mathscr{F}(M)$-bilinear, because it fails to be linear in the right component. Thus, ∇ is not a tensor. The connection ∇ is determined by its Christoffel symbols of second kind. They are defined as follows for each point $p \in M$.

Definition 1.8. *Let (M, ∇) be a manifold provided with an affine connection, $p \in M$ and $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ a basis of coordinate vector fields in a chart $(U, \mathbf{x})$ with $p \in U$. The* Christoffel symbols *are differentiable functions $\Gamma^n_{ij} : U \to \mathbb{R}$ such that:*

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^n} = \Gamma^n_{ij} \frac{\partial}{\partial x^n}. \tag{1.5}$$

Together with the affine connection, it is natural to define the covariant derivative, that is, a derivation of fields defined along a piecewise smooth curve, that is, $\frac{DX}{dt} = \nabla_{\gamma'} X$ when $X \in \mathfrak{X}(M)$. Nevertheless, in the case $X \in \mathfrak{X}(\gamma)$, which are vector fields defined in γ, that may not be induced by a field in M. See in [6, Chapter 2, prop. 2.2] the next result.

Proposition 1.9. *Let (M, ∇) be a differentiable manifold provided with an affine connection and $\gamma : (a, b) \to M$ a differential curve. Then, there exists a unique covariant derivative with the following properties:*

- $\frac{D}{dt}(X + Y) = \frac{DX}{dt} + \frac{DY}{dt}$.
- $\frac{DfX}{dt} = \gamma'(f) X + f \frac{DX}{dt}$.
- *If X comes from a vector field Y in M, then $\frac{DX}{dt} = \nabla_{\gamma'} Y$.*

Now, we have the ingredients to define parallel transport. The idea of parallel transport consists on transporting a vector field X along piecewise differentiable curves, so that the X "does not change" along this curve. This property means that the vector is parallel and can be formally expressed as $\frac{DX}{dt} = 0$. Besides, next proposition gives the uniqueness of parallel vector fields.

Proposition 1.10. *Let* (M, ∇) *be a differentiable manifold with an affine connection,* $\gamma : (a, b) \rightarrow M$ *a piecewise differentiable curve and a vector* v *in* $\gamma(t_0)$*. Then there exists a unique parallel vector field* $X \in \mathfrak{X}(\gamma)$ *such that* $X(t_0) = v$*.*

Proof. Let $\{x^1, \dots, x^n\}$ be a coordinate chart in $p = \gamma(t_0)$ such that $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$ and $X = \sum_i X^i \frac{\partial}{\partial x^i}$. We have that if X is parallel, it must fulfill the following equation:

$$\nabla_{\gamma'} X = \frac{dX^k}{dt} + \sum_{i,j} (\gamma^i)'(t) X^j \Gamma^k_{ij}(\gamma(t)) = 0 \quad k = 1, \dots, n. \tag{1.6}$$

The result in uniqueness is directly followed by differential equation theory, because this equation is a first order differential ordinary equation whose solution is unique for a given initial condition. □

Then, parallel transport shall be introduced as follows.

Definition 1.11. *Let* (M, ∇) *be a differentiable manifold with an affine connection,* $\gamma : [a, b] \rightarrow M$ *a curve and a vector* v *in* $\gamma(a)$*. Then, parallel transport is the application:*

$$\begin{array}{cccc} \tau : & T_{\gamma(a)}M & \rightarrow & T_{\gamma(b)}M \\ & v & \mapsto & \tau(v) \end{array}$$

where $\tau(v) = X(b)$ *and* X *is the parallel vector field along* γ *with initial condition* v*.*

Note that parallel transport define an isomorphism from $T_{\gamma(a)}M$ to $T_{\gamma(b)}M$. The linearity comes from the linear character of equation 1.6. The last point to consider is its bijectivity. It is injective because the unique parallel vector field that is the vector zero in $T_{\gamma(b)}M$ is the null vector, which characterizes injectivity. It is surjective because the two tangent spaces have the same dimension. We can also consider that $\tau_{\gamma^{-1}} = (\tau_\gamma)^{-1}$.

In case the curve is a loop, we may define a group of automorphisms in T_pM, the holonomy group, which will be explained in more detail in the second chapter.

Definition 1.12. *Let* (M, ∇) *be a differentiable manifold with an affine connection and let* p *be a point of* M*. Then, the* holonomy group *is the group of linear transformations that are a result of parallel translations of vectors along loops that may be either differentiable or piecewise differentiable with* p *as their basepoint, with the composition as its operation.*

Another important concept is that of geodesics. Classically, geodesics are the “straight” lines of a manifold.

Definition 1.13. $\gamma : (a, b) \rightarrow M$ *is a geodesic in a smooth manifold* (M, ∇) *if* $\frac{D\gamma'}{dt} = 0$ *along the curve* γ*.*

This definition means that the velocity field is parallel along the curve. Geodesics can be obtained solving a second order system of differential equations. If we write $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, and apply the chain rule:

$$\gamma''_k(t) + \sum_{i,j}^{n} \Gamma^k_{ij} \gamma'_i(t) \gamma'_j(t) = 0, \quad k = 1, \dots, n \tag{1.7}$$

The uniqueness of the solution of Cauchy problems of second order leads to the following result.

Theorem 1.14. *Let (M,∇) be a smooth manifold. For each $p \in M$ and $v \in T_pM$ there exists a unique geodesic γ in M such that $\gamma(0) = p$ and $\gamma'(0) = v$.*

Geodesics allow us to define the exponential map. This maps provides a local diffeomorphism of $\mathbb{R}^n$ and M, given by the uniqueness and existence of geodesics. Every $p \in M$ admits a neighborhood U such that the following application

$$\begin{array}{cccc} \exp\colon & U \subseteq T_pM & \to & \exp(U) \subseteq M \\ & v & \mapsto & \gamma(1,p,v) \end{array}$$

is a diffeomorphism. The notation $\gamma(t,p,v)$ is used to denote the point $\gamma(t)$ when γ is the unique geodesic starting at p with direction v.

Geodesics are related to the existence of a distance in Riemannian manifolds. If $L(\gamma) = \int_I \sqrt{g(\gamma',\gamma')}$ is the length of $\gamma : [0,1] \to M$, we can define:

$$d(p,q) = \inf_{\gamma:\gamma(0)=p,\gamma(1)=q} L(\gamma). \tag{1.8}$$

If there exists a curve such that $d(p,q) = L(\gamma)$, γ is a geodesic up to a reparameterization, see [6, Chapter 7].

Along this work, geodesic completeness will be studied.

Definition 1.15. *(M,g) is* geodesically complete *if all geodesics can be defined in $\mathbb{R}$.*

For Riemannian manifolds, there is a significant result, the Hopf-Rinow theorem, see [6, chapter 7], that relates geodesic completeness and metric completeness, among other results of the theorem.

Theorem 1.16. *Let (M,g) be a Riemannian manifold and $p \in M$. The following statements are equivalent.*

1. *$\exp_p$ is defined in the whole T_pM.*
2. *The closed and bounded sets are compact.*
3. *M is complete as a metric space.*
4. *M is geodesically complete.*
5. *There exists a sequence of compact sets $\{K_n\}_n$ such that $K_n \subset K_{n+1}$ and $\cup_n K_n = M$ such that any sequence of points $\{q_n\}_n$ with $q_n \notin K_n$, then $d(p,q_n) \to \infty$.*

In addition, any of the previous statements imply that for any $q \in M$, there exists a geodesic γ that joins p and q whose length $L(\gamma) = d(p,q)$.

In [13], we can find a result on completeness of manifolds.

Theorem 1.17. *Let M be a differentiable manifold. Then, M admits a Riemannian metric g that is complete.*

Levi-Civita connection

Given (M, g) a manifold with a metric, a natural question is if there exists an affine connection associated to this metric. The Levi-Civita connection is the affine connection that fulfills the following conditions:

- ∇ is said to be compatible with the metric, that is, the metric tensor is parallel for ∇, if, for every $X, Y, Z \in \mathfrak{X}(M)$:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{1.9}$$

- ∇ is a symmetric affine connection, that is the torsion:

$$Tor(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1.10}$$

  is equal to 0 for every $X, Y \in \mathfrak{X}(M)$. In this case, we say that the connection is symmetric

In [14], we can find a characterization of the Levi-Civita connection.

Theorem 1.18. *On a semi-Riemannian manifold M there is a unique symmetric connection ∇ compatible with the metric. ∇ is called the Levi-Civita connection and it is characterized by the Koszul formula:*

$$2g(\nabla_X Y, Z) = X\big(g(Y,Z)\big) + Y\big(g(X,Z)\big) - Z\big(g(Y,X)\big) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]). \tag{1.11}$$

Given a coordinate system, the Christoffel symbols can be calculated by using the above characterization.

Proposition 1.19. *For a coordinate system $\{x^1, \dots, x^n\}$, the Christoffel symbols are given by*

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} + \frac{\partial g_{jl}}{\partial x^i}\right). \tag{1.12}$$

Operators

The classical differential operators defined in $\mathbb{R}^n$ can be defined in any Riemannian or semi-Riemannian manifold. In the case of the Hessian, it is enough that a connection is defined.

Definition 1.20. *Let M be a differentiable manifold provided with an affine connection ∇, a function $f \in \mathscr{C}^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$, the* Hessian *of a function f is defined as:*

$$Hess f(X,Y) = (\nabla_X df)(Y) = X(Y(f)) - df(\nabla_X Y). \tag{1.13}$$

For the gradient, the metric has an essential role.

Definition 1.21. *Let (M, g) be a semi-Riemannian manifold, and $f \in \mathscr{C}^\infty$. The* gradient *of f ∇f is defined as the vector field that fulfills*

$$g(\nabla f, Y) = Y(f), \tag{1.14}$$

for $Y \in \mathfrak{X}(M)$.

1.2 Lorentzian Geometry

For the development of this section, [9, 14] are followed. Lorentzian manifolds have already been defined as differentiable n-manifolds whose metric has signature $p = n-1$ and $q = 1$. The relevance of Lorentzian manifolds in Physics comes from the development of Special and General Relativity. As a Lorentzian metric has signature $(p, q) = (n-1, 1)$, there are vectors with positive and negative norm, which implies the existence of degenerate vectors, that is, vectors with norm equal to zero. It is for its signature that Lorentzian manifolds are essential in the development of General Relativity.

Definition 1.22. *Let $v \in T_pM$, $v \neq 0$.*

- *If $g(v, v) > 0$ we say that v is a* spacelike *vector.*
- *If $g(v, v) < 0$ we say that v is* timelike *vector.*
- *If $g(v, v) = 0$ we say that v is* lightlike *vector.*
- *If $g(v, v) \leq 0$ we say that v is* causal *vector.*

All the vectos can be classified in terms of the first three kind of vectors. Nevertheless, there are several conventions to classify the zero vector. When we say that a vector is non-spacelike if it is causal or zero, nontimelike if it is spacelike, lightlike or zero and null if it is lightlike or zero. An important concept in Lorentzian geometry is that of timelike cones defined in the tangent space of each point.

Proposition 1.23. *Let M be a Lorentzian manifold and $p \in M$. The subset of timelike vectors of T_pM has two connected parts, each one is called timelike cone.*

Proof. Taking $\{e_1, \ldots, e_n\}$ any orthonormal basis of T_pM and a vector $v = \sum_{i=1}^{n} a^i e_i$, the vector is timelike if $a^1 \neq 0$ and $|a^1| > \sqrt{(a^2)^2 + \ldots + (a^n)^2}$. With this definition, the subset of timelike vectors has two connected parts. □

The causal and lightlike cones are studied in the same way. Cones are useful to introduce the notions of future and past in Lorentzian geometry.

Definition 1.24. *Let (M, g) be a Lorentzian manifold. A* time orientation in $p \in M$ *is the choice of one timelike cone, which is called the* future of p. *The other cone is called the* past of p.

A time orientation in M *is a map τ that assigns a timelike cone $\tau_p \subset T_pM$ to each $p \in M$ continuously. This means that in each $p \in M$ there is some open neighborhood U_p such that a cone $X_q \in \tau_q$ for each $q \in U_p$. (M, g) is called* time orientable.
A spacetime *is a connected time orientable Lorentzian manifold endowed with a time orientation.*

Proposition 1.25. *A Lorentzian manifold (M, g) is time orientable if and only if it admits a globally defined timelike vector field $X \in \mathfrak{X}(M)$.*

Proof. If (M,g) is time orientable, take $\{(U_\alpha,\lambda_\alpha)\}$ partition of the unity of the Lorentzian manifold. Each point is in a finite number of domains U_{α_i}. Because of the previous definition, take a future-directed timelike vector field in each domain X_i. Because of convexity of the timelike cone contains any finite sum of these timelike cones. The field $X = \sum_\alpha \lambda_\alpha X_\alpha$ is the required timelike vector. Conversely, take τ_p as the time orientation of X_p. □

Lorentzian manifolds are endowed with an special topology, the Alexandrov topology. We introduce some notation in a Lorentzian manifold (M,g):

- Let $p,q \in M$, we say $p \ll q$ if there is a smooth future directed timelike curve from p to q.
- Let $p,q \in M$, we say $p \le q$ if there is a smooth future directed nonspacelike curve from p to q or if $p = q$.

With this concepts, we can define the chronological and causal pasts and futures of $p \in M$.

- Chronological future: $I^+(p) = \{q \in M : p \ll q\}$.
- Chronological past: $I^-(p) = \{q \in M : q \ll p\}$.
- Causal future: $J^+(p) = \{q \in M : p \le q\}$.
- Causal past: $J^-(p) = \{q \in M : q \le p\}$.

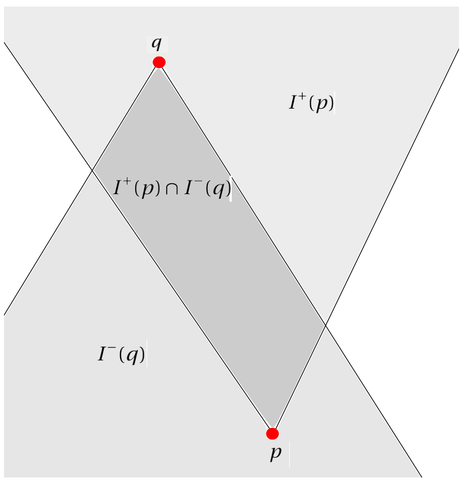


Figure 1.1: Open set of the basis of the Alexandrov topology.

In the topology of M as a differential manifold, the chronological pasts and futures are open subsets of M. This is because in T_pM the set of vectors $V^+ = \{v \in T_pM : g(v,v) < 0, v \text{ is future directed}\}$ is open. Considering that the exponential map is continuous and locally a diffeomorphism $\exp_p(V^+)$ is open.

Definition 1.26. *The* Alexandrov topology *is the topology generated by the open sets $I^+(p) \cap I^-(q)$ for $p, q \in M$.*

As the sets on the form $I^+(p) \cap I^-(q)$, see figure 1.1, are open in the manifold topology, the topology of the manifold can either coincide with the Alexandrov topology or be finer.

1.2.1 The causal ladder

In this section, we introduce the ambient of work of the physically viable spacetimes, even if in our work we will work mainly with generic Lorentzian manifolds. Lorentzian geometry usually serves to define physical spaces, under some restrictions. For example, a timelike curve that intersects itself does not match with the physical description of spacetimes, as it leads to paradoxes. In this section, we study the causal ladder in which some characteristics of increasing restrictiveness are added to spacetimes. We follow figure 1.2 to structure the explanation. This section is based on the references [12, 9].

We start denifing chronological and causal spacetimes.

Definition 1.27. (M, g) *is a* chronological *spacetime if it does not contain closed timelike curves. If it does not contain closed causal geodesics it is a* causal *spacetime.*

Causal spacetimes, by definition, are chronological spacetimes as well. A stronger type of spacetime is distinguishing spacetimes.

Definition 1.28. *Let (M, g) be a spacetime and $p, q \in M$. We say that M is a* distinguishing *spacetime if it satisfies that $I^+(p) = I^-(q)$ if and only if $p = q$.*

In a distinguishing spacetime, it is not possible that a causal geodesic is closed, so distinguishability implies causality.

Definition 1.29. *A spacetime (M, g) is said to be* strongly causal *if and only if for any $p \in M$ and for any neighborhood U of p there exists $V \subset U$ such that any causal curve that leaves V does not return to it.*

An equivalent condition is that a curve with both endpoints in V is completely contained in U. Another characterization of strongly causal spacetimes comes from its Alexandrov topology.

Theorem 1.30. *A spacetime is strongly causal if and only if the Alexandrov topology is equal to the topology of M.*

In the definition, we use a basis of neighborhood of metrics. The neighborhood of metrics are connected subsets in Lor(M), the space of Lorentzian metrics, in which their coefficients are close to those of g. A formal definition is given in Chapter 3.

Definition 1.31. *A spacetime (M, g) is* stably causal *if there exists a neighborhood of metrics of g such that M is causal for every metric in U.*

But the most common characterization of stable spacetimes follows. For the next step in the ladder, we will define a time function to give a characterization of stably causal spacetimes. A time function is a continuous $t : M \to \mathbb{R}$ strictly increasing in future-directed causal curves.

Globally hyperbolic
⇓
Causally simple
⇓
Causally continuous
⇓
Stably causal
⇓
Strongly causal
⇓
Distinguishing
⇓
Chronological

Figure 1.2: The causal ladder.

Theorem 1.32. (M,g) *is a* stably causal *spacetime if and only if it admits a time function.*

The next two kinds of spacetimes require the notion of time function for their definition. We will need to understand the concept of admissible measures. Let h be an auxiliary Riemannian metric such that the volume of M is finite, then the volume function that comes from this measure is the admissible measure m. With this measure, the functions t^+, t^- are defined as $t^+(p) = -m(I^+(p))$ and $t^-(p) = -m(I^-(p))$. These functions are non-decreasing and continuous in future-directed causal curves by definition, and thus, in case they are increasing, they can serve as time functions.

Definition 1.33. (M,g) *is a* causally continuous *spacetime if t^+ and t^- are time functions.*

Intuitively, this means that the chronological future and past vary continuously with the point. Notice, however, that the causal future and past are not always closed (a simple example appears by removing a point in Lorentz-Minkowski). So, the next step ensures this property.

Definition 1.34. (M,g) *is* causally simple *if it is causal and $J^+(p)$ and $J^-(p)$ are the closures of $I^+(p)$ and $I^-(p)$ for all $p \in M$.*

Finally, we introduce the globally hyperbolic spacetimes.

Definition 1.35. (M,g) *is* globally hyperbolic *if $J^+(p) \cap J^-(q)$ is compact for every $p,q \in M$.*

Next theorem characterizes globally hyperbolic spacetimes with Cauchy hypersurfaces. Let S be a hypersurface of (M,g), then it is a Cauchy hypersurface if it is crossed by any timelike curve only once. This is a result by [7].

Theorem 1.36. (M,g) *is globally hyperbolic if and only if it admits a Cauchy hypersurface.*

All of these kinds of spacetimes are well ordered in the sense that the above levels imply the lower ones. In figure 1.2, if a level is above another, this level imply the lower one (as well as the rest levels below this one). See [12] to see the precise proofs for this property.

Chapter 2

Precompact holonomy and completeness

Along this chapter, it will not matter if the metric is Lorentzian or Riemannian because the target of study is the parallel transport and its holonomy group. This concept is basic in Differential geometry, but it has not been studied in the Degree or in the Master. In the first section, we study some properties of the holonomy group. We start proving that the holonomy group is well-defined, that is, that it is actually a group with the composition operation. When we work with the Levi-Civita connection for some metric, we can ensure that the holonomy group is a subgroup of the orthogonal group of the correspondent signature. In the second section, there is an interesting result that relates precompact holonomy and completeness in compact manifolds. This result is independent of the metric, but, as we will see, the conditions are always fulfilled for compact Riemannian manifolds. Even if this result is rather simple, it was obtained recently and will serve as a basis for the further development.

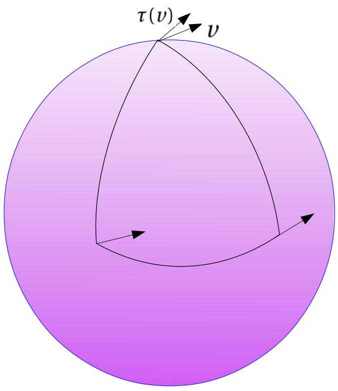
2.1 The holonomy group

For the development of this section, we follow [5]. The holonomy group was introduced in definition 1.12. As parallel transport is an isomorphism between the two tangent planes in the endpoints of the curve, if the curve is a loop, both endpoints are the same and parallel transport is an automorphism. Therefore, the holonomy group for any $p \in M$ and for any connection ∇ is a subset of $Aut(T_pM)$, isomorphic to $GL(n, \mathbb{R})$. In the definition of holonomy group, we stated that $Hol_p(\nabla)$ is a group, but the group structure needs to be proven. The operation of this group is the composition, which, as we will see, is related to the concatenation of paths.

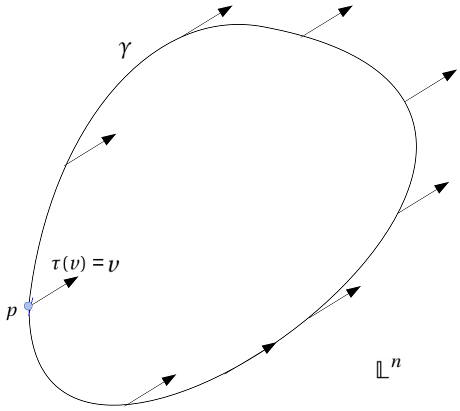
Definition 2.1. *Let* $\alpha : [0,1] \to M, \beta : [0,1] \to M$ *be two curves such that* $\alpha(1) = \beta(0)$*. We call* $\alpha * \beta$ *the* concatenated curve of α *and* β*, which is defined as:*

$$\alpha * \beta = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \tag{2.1}$$

Note that the definition is given for curves defined in $[0,1]$. For any interval $[a,b]$ we can change variable such that the curve has the same trace but it is defined in $[0,1]$. We want to



(a)



(b)

Figure 2.1: Parallel transport of a loop of the sphere with the metric induced by the Euclidean metric in $\mathbb{R}^3$ and parallel transport of a loop in $\mathbb{L}^n$.

see that the composition is well defined by analyzing the element given by the composition of two transformations in $Hol_p(\nabla)$.

Proposition 2.2. *Let* τ_α, τ_β *be two automorphisms in* $Hol_p(\nabla)$*. Then,* $\tau_\beta \circ \tau_\alpha = \tau_{\alpha * \beta}$*.*

Proof. Let $v \in T_pM$ be a vector. Then, $\tau_\alpha(v)$ is the image of v by the parallel transport of α. $\tau_\beta(\tau_\alpha(v))$ is the image of $\tau_\alpha(v)$ by its parallel transport along β. Secondly, $\tau_{\alpha*\beta}$ is the result of applying first the parallel transport of α, τ_α and secondly the parallel transport of β, leading to $\tau_{\alpha*\beta} = \tau_\beta \circ \tau_\alpha$. □

By the last proposition, it is immediate that the composition is well-defined and that $\circ$ is a closed operation. Given that c_0 is the constant curve in p, we have that $\tau_{c_0} = id_{T_pM}$, the neutral element. In addition, $\tau_\gamma^{-1} = \tau_{\gamma^{-1}}$. With the existence of neutral and inverse element, and proposition 2.2, we have that the holonomy group in p is well-defined as a group.

Another aspect to consider is the relation among holonomy groups in the different points of the manifold.

Proposition 2.3. *Let* (M, ∇) *be a connected manifold and* $p, q \in M$*. Let* $\gamma : [a, b] \to M$ *be a curve such that* $\gamma(a) = p, \gamma(b) = q$*. Then,*

$$Hol_p(\nabla) = \tau_\gamma^{-1} \circ Hol_q(\nabla) \circ \tau_\gamma \tag{2.2}$$

Proof. Let α be a piecewise differentiable loop in q. Then, we can define a loop in p $\bar{\alpha} = \gamma * \alpha * \gamma^{-1}$. We have that $\tau_{\bar{\alpha}} = \tau_\gamma^{-1} \circ \tau_\alpha \circ \tau_\gamma$ is an automorphism in $Hol_p(\nabla)$, which leads to $\tau_\gamma^{-1} \circ Hol_q(\nabla) \circ \tau_\gamma \subseteq Hol_p(\nabla)$.

On the other hand, let $\bar{\alpha}$ be a loop in p. $\alpha = \gamma^{-1} * \bar{\alpha} * \gamma$ is a loop in q. And we have that $\bar{\bar{\alpha}} = \gamma * \gamma^{-1} * \bar{\alpha} * \gamma * \gamma^{-1} = \gamma * \alpha * \gamma^{-1}$ is a loop in p such that $\tau_{\bar{\bar{\alpha}}} \in \tau_\gamma^{-1} \circ Hol_q(\nabla) \circ \tau_\gamma$. Besides,

$$\tau_\alpha = \tau_\gamma \circ \tau_\gamma^{-1} \circ \tau_\alpha \circ \tau_\gamma \circ \tau_\gamma^{-1} = \tau_{\bar{\bar{\alpha}}}.$$

Therefore, $Hol_p(\nabla) \subseteq \tau_\gamma^{-1} \circ Hol_q(\nabla) \circ \tau_\gamma$. □

With this proposition, we have got that that the holonomy groups of each connected component are the same up to an isomorphism, which depends on the curve chosen to go from one point to another. Hence, we can talk about the holonomy group of a connected manifold and it is not necessary to specify the point. As two examples of holonomy groups, in figure 2.1a, the holonomy group of the sphere is isomorphic to $SO(2)$. In $\mathbb{L}^n$ for any $n \in \mathbb{N}$, see figure 2.1b, the holonomy group is $\{id_{\mathbb{R}^n}\}$.

In the previous examples, the Levi-Civita connection of a metric has been considered. In these cases, an additional property can be studied; parallel transport is an isometry.

Theorem 2.4. *Let (M,g) be a differentiable manifold of arbitrary signature and $\gamma : [a,b] \to M$ any curve joining p and q. τ_γ is an isometry.*

Proof. Let X, Y be any two parallel vectors in γ. Defining $f(t) = g(X(t), Y(t))$:

$$f'(t) = g\left(\frac{DX}{dt}, Y\right) + g\left(\frac{DY}{dt}, X\right) = 0.$$

Thus, $g(X,Y)$ is constant along the curve and τ_γ is an isometry because if $X(a) = v$, $Y(a) = w$:

$$g(\tau_\gamma(v), \tau_\gamma(w)) = g(X(b), Y(b)) = g(X(a), Y(a)) = g(v,w).$$

□

Because of this theorem, we have that when ∇ is the Levi-Civita connection of a Riemannian manifold, in a coordinate chart $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$ where the vector fields are orthonormal in p, $Hol_p(\nabla) \leq O(n)$, the orthogonal group of dimension n. If the metric has arbitrary signature (p,q), $Hol_p(\nabla) \leq O(p,q)$, the orthogonal group for this signature. Note that changing the basis of vectors, so does the matrix representation of $Hol_p(\nabla)$, that is, $Hol_p(\nabla)$ in the basis $\left\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\}$is represented as a conjugate subgroup of when it is represented in the orthonormal basis $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$. The orthogonal group of signature (p,q) is defined as:

$$O(p,q) = \left\{M^T \in GL(n,\mathbb{R}) : M \cdot I_{p,q} \cdot M = I_{p,q}\right\}. \tag{2.3}$$

where

$$I_{p,q} = \begin{pmatrix} 1 & 0 & & & \cdots & & 0 \\ 0 & & \overset{p}{\ddots} & & & & \\ & & & 1 & & & \vdots \\ \vdots & & & & -1 & & \\ & & & & & \overset{q}{\ddots} & 0 \\ 0 & & & \cdots & & 0 & -1 \end{pmatrix}$$

In the next section, we will work with manifolds that have precompact holonomy. $GL(n,\mathbb{R})$ is a topological space, that can be seen as a subset of the square matrices, homeomorphic to $\mathbb{R}^{n^2}$. Then, a compact subgroup of $GL(n,\mathbb{R})$ is a compact subset. As we have seen that compact subgroups are compact subsets of $\mathbb{R}^{n^2}$, they can be characterized as:

1. If $\{M_n\}_n$ is a sequence of matrices in G, it has a convergent subsequence to some $M \in G$.

2. G is closed and there exists a constant $C > 0$ such that $|M_{ij}| \leq C$ for all $M \in G$.

We say that a group is precompact if its closure is compact.

As an example of a compact group, we have the Euclidean orthogonal group $O(n)$, which is closed, as it is $f^{-1}(I)$ of the following application:

$$\begin{array}{rccc} f: & GL(n,\mathbb{R}) & \to & GL(n,\mathbb{R}) \\ & M & \mapsto & M^T \cdot M \end{array}$$

It is compact because for every $i = 1,\dots,n$, $M_{i1}^2 + \ldots + M_{in}^2 = 1$, which leads to $|M_{ij}| \leq 1$. Applying this to the holonomy groups of Riemannian manifolds, we have that Riemannian manifolds always have precompact holonomy, because their holonomy groups are always subgroups of $O(n)$.

Nevertheless, if $1 \leq p, q \leq n-1$, the orthogonal group $O(p,q)$ is not compact. Taking the orthogonal group $O(1,1)$, we have that $\begin{pmatrix} \sinh(t) & \cosh(t) \\ \cosh(t) & \sinh(t) \end{pmatrix} \in O(1,1)$ for every $t \in \mathbb{R}$, whose coefficients are not bounded. For the orthogonal group with general signature, we have that the matrices:

$$\begin{pmatrix} I_{p-1} & & & \\ & \sinh(t) & \cosh(t) & \\ & \cosh(t) & \sinh(t) & \\ & & & I_{q-1} \end{pmatrix} \in O(p,q),$$

for $t \in \mathbb{R}$, which leads to the unboundedness of the orthogonal group, and thus, its non compactness. In the particular case of Lorentzian manifolds, we cannot ensure that they have precompact holonomy as the holonomy groups are subgroups of $O(n-1,1)$. But, Lorentzian manifolds can have precompact holonomy, because $O(n-1,1)$ can have compact subgroups, take for example $\{Id\}$ or $O(n-1)$.

An important property of precompact holonomy groups is that if a sequence of vectors lies in a compact set, the transformed vectors by elements of $Hol_p(\nabla)$ lie in a compact set as well.

Lemma 2.5. *Let $\{v_n\}_n \subset K \subset T_pM$ a sequence of vectors and K a compact set in T_pM and let $\{\tau_n\}_n$. If $Hol_p(\nabla)$ is precompact, the sequence $\{\tau_n(v_n)\}_n$ lies on a compact set $\tilde{K}$ of T_pM.*

Proof. Recall that $Hol_p(\nabla)$ can be represented as a subgroup of the group of matrices $GL(n,\mathbb{R})$ for some fixed basis and, as it is precompact, there is $C > 0$ such that the coefficients of $M \in Hol_p(\nabla)$ are bounded by C, $|M_{ij}| \leq C$. On the other hand, the sequence of vectors lies in a compact set, therefore, there is $\bar{C}$ a constant such that $||v_n||_\infty \leq \bar{C}$ for $n \in \mathbb{N}$. Then, $||\tau_n(v_n)||_\infty \leq \tilde{C}$ for some $\tilde{C}$, and the sequence $\{\tau_n(v_n)\}_n$ lies in a compact set. □

In the next section, the precompact holonomy condition is related to the geodesic completeness in the manifold, when the manifold is compact.

2.2 Holonomy and completeness

Next result was developed in [8]. It offers a way to ensure completeness by studying compactness and the holonomy group. To prove this theorem, we will need to use lemma 2.6.

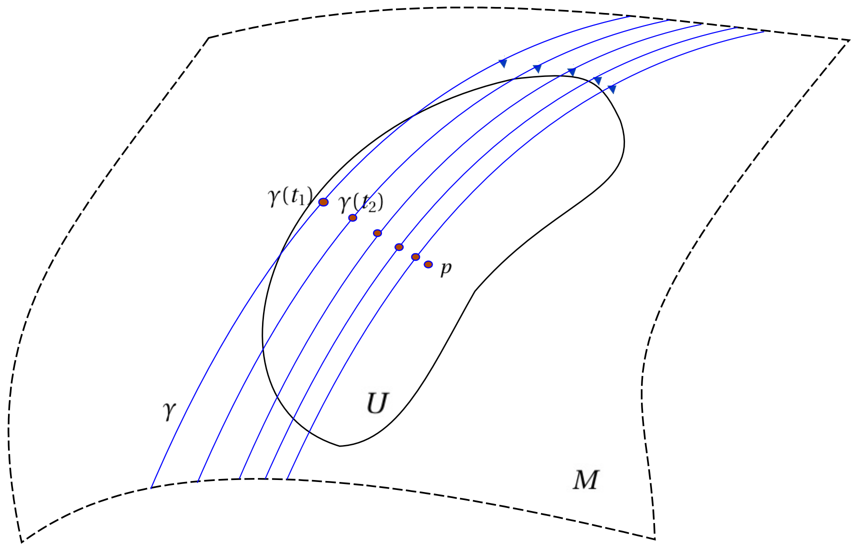


Figure 2.2: The geodesic γ through a convex normal neighborhood U, and the sequence of points $\{\gamma(t_n)\}_n$ converging to p.

Lemma 2.6. *Let $\gamma : (a,b) \to M$ be an inextendible geodesic with $b < \infty$. It $\{t_n\}_n$ is a sequence in (a,b) such that $\{\gamma(t_n)\}_n$ and $\{\gamma'(t_n)\}$ converge in M and TM, respectively, then, γ can be extended to p.*

Proof. For this proof we will follow [14]. In chapter 3, proposition 28, all the curves in TM obtained as the velocity γ' of a geodesic can be regarded as the integral curve of the a vector field X on TM ($X \in \mathfrak{X}(TM)$). Secondly, we use a result, lemma 56 of chapter 1, that states that if $\alpha : (a,b) \to M$ is an integral curve and $\{\alpha(t_n)\}$ converges for a sequence $\{t_n\}_n$ convergent to b, the integral curve can be extended beyond $z = \lim_{n\to\infty} \alpha(t_n)$. Taking γ' in TM, as it is the velocity of a geodesic, we can regard $\alpha(t) = \gamma'(t)$ in (a,b) as an integral curve of $X \in \mathfrak{X}(TM)$. The sequence $\{\gamma(t_n)\}_n$ converges to $p \in M$ and the sequence $\{\gamma'(t_n)\}$ converges to some $v \in T_pM$. Thus, $\alpha = \gamma'$ can be extended beyond b as an integral curve of X, that is, γ can be extended as a geodesic beyond b. □

Theorem 2.7. *Let (M,∇) be a compact manifold provided with a linear connection ∇ and let $p \in M$. If the holonomy group in p is precompact (in $GL(T_pM)$), then (M,∇) is geodesically complete.*

Proof. We will assume that there is an incomplete geodesic arriving to a contradiction with the precompact holonomy condition. Let $\gamma : [0,b) \to M$ be an incomplete inextendible geodesic. As M is compact, we can choose a sequence $\{t_n\}_n$ convergent to b such that the sequence $\{\gamma(t_n)\}$ converges to a point $p \in M$. Because M is a smooth manifold, let $U \subset M$ be an open starshaped normal neighborhood of p. Due to convergence, we may assume that $\{\gamma(t_n)\}_n \subset U$, as it can be seen in figure 2.2. Defining β_n as the radial geodesic from p to $\gamma(t_n)$ and $\alpha_n = \beta_n^{-1} * \gamma_{|[t_1,t_n]} * \beta_1$ as the loops in p so that it goes to $\gamma(t_1)$, goes along γ until $\gamma(t_n)$

and comes back to p along a radial geodesic. We call $v = \tau_{\beta_1^{-1}}(\gamma'(t_1))$, that is, the velocity of the curve at $\gamma(t_1)$ transported parallelly to p. For the rest of the proof, we will work with that vector. Recalling that γ is a geodesic, one can define:

$$v_n := \tau_{\alpha_n}(v) = \tau_{\beta_n^{-1}} \circ \tau_{\gamma_{|[t_1,t_n]}} \circ \tau_{\beta_1}(\tau_{\beta_1^{-1}}(\gamma'(t_1))) = \tau_{\beta_n^{-1}}(\gamma'(t_n)).$$

As $Hol_p(\nabla)$ is precompact, that is, $\overline{Hol_p(\nabla)}$ is compact, the sequence of vectors $\{v_n\} \subset T_pM$ lies in a compact set K. To find the contradiction, we need to see that the sequence of velocities $\{\gamma'(t_n)\}$ lies on a compact set of TM, implying that it contains a convergent subsequence and yielding the contradiction by lemma 2.6. Let $\tilde{K} \subset \exp^{-1}(U)$ be a compact starshaped neighborhood of 0 in T_pM such that $\rho_u : [0,1] \to U$ is the radial geodesic with initial velocity $u \in \tilde{K}$. Let ξ be the map:

$$\begin{array}{rccc} \xi: & \tilde{K} \times K & \to & TM \\ & (u, v) & \mapsto & \tau_{\rho_u}(v) \end{array}$$

By its construction, ξ is a continuous map. Thus, its image $\bar{K} = \xi(\tilde{K}, K) \subset TM$ is compact. Furthermore, this set contains $\{\gamma'(t_n)\}_{n \geq n_0}$ for a sufficiently large n_0, because $\gamma(t_n) \in \exp(\tilde{K}) = U$ due to its convergence to p. As $\rho_{\exp^{-1}(\gamma(t_n))} = \beta_n$ because they are radial geodesics with the same endpoints, $\gamma'(t_n) = \xi(\exp^{-1}(\gamma(t_n)), v_n)$,there is a convergent subsequence of velocities and thus, we find the contradiction with incompleteness. □

Observation 2.8. Notice that compact Riemannian manifolds always fulfill the conditions of the previous theorem. This is because the holonomy group is always a subgroup of a compact group, and therefore, its closure is always compact. Nevertheless, completeness was also known by the Hopf-Rinow theorem.

Chapter 3

Stability of completeness and incompleteness

In this chapter, we study stability of completeness and incompleteness rather than completeness and incompleteness of semi-Riemannian manifolds themselves. In the first section, some example of incomplete Lorentzian manifolds are explained. Then, the concept of stability is introduced, as well as a topology of the space $\mathrm{Lor}(M)$ and $\mathrm{Pseudo}(M)$ of Lorentzian and semi-Riemannian metrics of a manifold M. In the third section, the stability of incompleteness is studied, whereas the stability of completeness is studied in the last one.

3.1 Some examples of incomplete Lorentzian manifolds

It is widely known that completeness in Riemannian manifolds can be studied using Hopf-Rinow theorem. However, this theorem cannot be applied to semi-Riemannian manifolds. In fact, there is not a theorem as strong as Hopf-Rinow theorem to study completeness. Hence, the concepts of metric completeness and geodesic completeness are separated. As a remark, even compact Lorentzian manifolds can fail to be geodesically complete whereas they are complete for any distance. A good example to illustrate that this can happen is the Clifton-Pohl torus, see [15].

Example 3.1. *(Clifton-Pohl torus)* The Clifton-Pohl torus is defined as an example of a compact Lorentzian manifold that is incomplete. Let $S = \mathbb{R}^2 \setminus \{0\}$ endowed with the metric $g(x, y) = \frac{2dxdy}{x^2+y^2}$.To define the Clifton-Pohl torus, take $G = \left\{f^n, n \in \mathbb{Z}\right\}$ the group generated by $f(x, y) = (2x, 2y)$, which are isometries because $f^*g = g$. G is properly discontinuous, because for any $p \in S$, there exists a neighborhood U such that there is a unique $\bar{f} \in G$ with $\bar{f}(U) \cap U \neq \emptyset$, which is the identity. Then, the Clifton-Pohl torus is defined as the quotient space $T_{C-P} = S/G$, which is compact. First, we study a geodesic that is incomplete in S. The Christoffel symbols for its metric are $\Gamma^1_{11} = \frac{2x}{x^2+y^2}$, $\Gamma^2_{22} = \frac{2y}{x^2+y^2}$ and $\Gamma^k_{ij} = 0$ for the rest index triplets. Therefore, the geodesic equations are:

$$\begin{aligned} x'' &= \tfrac{2x}{x^2+y^2}(x')^2 \\ y'' &= \tfrac{2y}{x^2+y^2}(y')^2 \end{aligned}$$

Taking the geodesic $\gamma : (-\infty, 1) \to S$ defined as $\gamma(t) = \left(\frac{1}{1-t}, 0\right)$, incompleteness in the forward direction can be seen because γ is not defined in $\mathbb{R}$, as it cannot be defined in $t = 1$.

Besides, the projection $\pi : S \to T_{C-P}$ is a local isometry because T_{C-P} is S under the action of isometries. The geodesic $\bar{\gamma} = \pi \circ \gamma : (-\infty, 1) \to T_{C-P}$ is incomplete as γ was in S.

Next, we characterize the existence of incomplete (necessarily lightlike) geodesics by means of a simple criterion (we follow [2]).

Lemma 3.2. *Let (M, g) be a Lorentzian manifold and $\gamma : (a, b) \to M$ an inextendible null geodesic such that $\gamma(0) = \gamma(1)$ and $\gamma'(1) = \lambda\gamma'(0)$ for some $\lambda > 0$. If $\lambda = 1$, the geodesic λ is complete and γ is incomplete otherwise. In particular, if $\lambda > 1$, γ is incomplete in the forward direction and if $\lambda < 1$, in the reverse one.*

Proof. If $\lambda = 1$, the proof is trivial since γ and γ' are periodic. In case $\lambda > 1$, we have that the velocity increases in a factor λ in each period. We define $\alpha(t) = \gamma(\lambda t)$, a geodesic that fulfills $\alpha(\lambda^{-1}) = \gamma(1) = \gamma(0) = \alpha(0)$ and $\alpha'(0) = \lambda\gamma'(0) = \gamma'(1)$, so it is a null closed geodesic and $\alpha(t) = \gamma(1 + t)$. Its first loop $\alpha[0, \lambda^{-1}]$ corresponds to the second loop of γ, $\gamma[1, 2]$. Thus, a countably number of loops of γ increase the affine parameter of γ by $\sum_{n=0}^{\infty} \lambda^{-n} = \frac{1}{1-\lambda^{-1}}$. This is a finite number because $\lambda > 1$, and it is an upper bound for the interval of definition, in fact, as the interval of definition is maximal, $b = \frac{1}{1-\lambda^{-1}} < \infty$. Therefore, the geodesic is incomplete in the forward direction. Nevertheless, in the reverse direction, the affine parameter changes in an infinite number of loops by $\sum_{n=0}^{\infty} \lambda^{n} = \infty$.

If $\lambda < 1$, the incompleteness is proven equally, studying the loops in the reverse direction of γ. □

Notice that the previous lemma stands only for null geodesics. This is because for space and timelike geodesics $g(\gamma', \gamma') \neq 0$ is a constant and g is positive and negative definite for this vectors, respectively. Therefore, if the velocity increases by a factor $\lambda > 0$, $g(\lambda\gamma', \lambda\gamma') = \lambda^2 g(\gamma', \gamma')$, and equality to $g(\gamma', \gamma')$ stands only for $\lambda = 1$. In case the geodesic is null, $g(\lambda\gamma', \lambda\gamma') = 0$ remains constant for any λ.

Another typical example of incomplete Lorentzian metrics is Misner cylinder, see [9].

Example 3.3. *(Misner cylinder)* Misner cylinder is defined on $M_0 = (0, \infty) \times \mathbb{R}$ with the metric $g(x, y) = dx \otimes dy + dy \otimes dx = 2dxdy$. We define the group of isometries $G = \{\phi^k : \phi(x, y) = \left(2x, \frac{1}{2}y\right)\}$. This is clearly a group of isometries, given any vector (a, b) with $g((a, b), (a, b)) = 2ab$, $g(\phi(a, b), \phi(a, b)) = 2(2a)\left(\frac{1}{2}b\right) = g((a, b), (a, b))$. Then, Misner cylinder is $M = ((0, \infty] \times \mathbb{R})/G$, and can be represented in $[1, 2] \times \mathbb{R}$, identifying $(1, 2y) \sim (2, y)$ as it can be seen in figure 3.1. In the Misner cylinder we can find an example of an incomplete closed geodesic. Let $\gamma(t) = (t, 0)$ be the horizontal straight line starting in $\gamma(0) = (1, 0)$ with velocity $\gamma'(0) = (1, 0)$, the red vector in figure 3.1. This is a null geodesic because $g(\gamma', \gamma') = 0$ and it comes from the geodesic in M_0 defined as the horizontal straight line $\tilde{\gamma}$ with constant velocity $\tilde{\gamma}'(t) = (1, 0)$. However in M, let $t_1 > 0$ and $t_2 = 2t_1$, $\gamma'(t_2) = 2^{-1}\gamma'(t_1)$. By the previous lemma, the geodesic is incomplete in the reverse direction.

3.2 Fine $\mathscr{C}^r$ topologies of metrics

Along this chapter, some results in completeness and incompleteness of Lorentzian manifolds are given. The aim of study is the stability of completeness and incompleteness, fol-

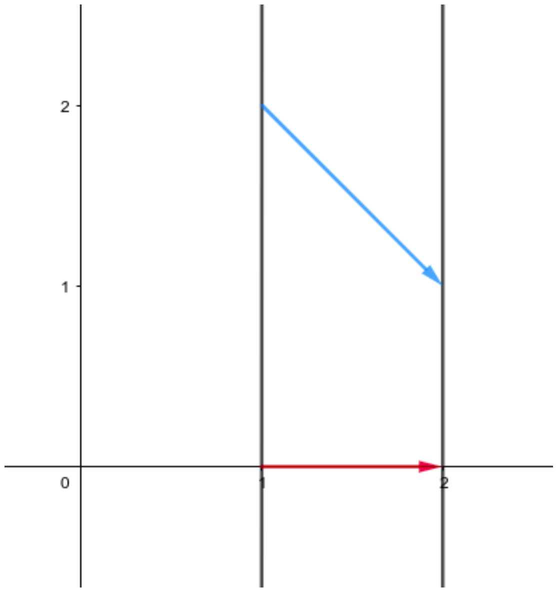


Figure 3.1: In this figure, there is a representation of the Misner cylinder. The blue and red vector indicates two of the points that are identified.

lowing [2]. The concept of stability in this context must be explained. We say that a property in a Lorentzian manifold (M, g) is stable if there exists a neighborhood of close metrics U of g such that if $\bar{g} \in U$, $(M, \bar{g})$ fulfills the property. But, to study stability, we need to know which metrics are close to others and a topology to define the neighborhoods.

A collection of topologies in the space of $\mathrm{Lor}(M)$, that is, Lorentzian metrics of a manifold M, is the fine $\mathscr{C}^r$ topologies. They are defined in a locally finite covering $\{B_i\}_i$ and such that for each i $\overline{B_i}$ lies inside a coordinate chart. Let M be a differential manifold. Two Lorentzian metrics $g, \bar{g}$ are said to be $\delta : M \to (0, \infty)$ close in the $\mathscr{C}^r$ topology if for every $p \in M$ and for every B_i such that $p \in B_i$, the coefficients for the metric in p and its derivatives up to order r are closer than $\delta(p)$, that is,

$$\left| \frac{\partial^m g_{ij}}{\partial x^{k_1} \overset{m}{\cdots} \partial x^{k_m}} - \frac{\partial^m \bar{g}_{ij}}{\partial x^{k_1} \overset{m}{\cdots} \partial x^{k_m}} \right| < \delta(p) \quad \text{for } i, j, k_\ell = 1, \ldots, n, \quad \ell = 0, \ldots, m \quad \text{and } m = 0, \ldots, r.$$

This is denoted as $|g - \bar{g}|_r < \delta$. This concept leads to the definition of $\mathscr{C}^r$ neighborhoods of metrics.

Definition 3.4. *Let (M, g) be a Lorentzian manifold, we say that $U(g)$ is a $\mathscr{C}^r$* neighborhood of g in $\mathrm{Lor}(M)$ *if $U(g) = \{\bar{g} \in Lor(M), |g - \bar{g}| < \delta\}$, where $\delta : M \to (0, \infty)$.*

The fine $\mathscr{C}^r$ topology has these open sets as its basis. Because of the definition, the fact that $\mathscr{C}^r \subset \mathscr{C}^{r'}$ if $r < r'$ follows. The topology of interest along this work is the fine $\mathscr{C}^1$ topology, because in the study of geodesic completeness, it is precise to use the metric tensor as well as its derivatives.

In case the metric is Riemannian or negative definite, we have stability of both completeness and incompleteness for each $r \geq 0$. This is because we can take any neighborhood $U(g)$ of metrics of g such that $C_1 < \frac{\bar{g}(v,v)}{g(v,v)} < C_2$. The length of any segment of geodesic fulfills $\sqrt{C_1} L_g(\gamma) \leq L_h(\gamma) < \sqrt{C_2} L_g(\gamma)$.

On the other hand, if the metric is semi-Riemannian, we can define neighgorhood of metrics in Pseudo(M), the set of semi-Riemannian metrics. In fact, the results of the next sections work in any semi-Riemannian manifold, Lorentzian or with an arbitrary signature. The next results show the conditions that an incomplete or complete semi-Riemannian manifold fulfills in order to ensure the existence of a neighborhood of incomplete, respectively complete, metrics for the differentiable manifold. The results of next sections were developed by Beem and Erlich. They establish sufficient conditions for complete and incomplete semi-Riemannian manifolds to ensure the existence of a $\mathscr{C}^1$ neighborhood of incomplete (resp. complete) metrics in Pseudo(M).

3.3 Stability of incompleteness

The purpose of this section is to introduce a result that ensures that an incomplete Lorentzian manifold (M, g) may be stable, in the sense that there exists $U(g)$ a $\mathscr{C}^1$ neighborhood of metrics such that if $\bar{g} \in U(g)$, $(M, \bar{g})$ has an incomplete geodesic as well. Next example shows that an incomplete Lorentzian manifold has close metrics that are incomplete as well. We remark that this example does not show that incompleteness is stable, but that there can be Lorentzian metrics close to an incomplete one that are incomplete.

Example 3.5. Let $\mathbb{R}^2$ be endowed with the metric $g(x, y) = 2dxdy + \tau(x)dy^2$, where $\tau : \mathbb{R} \to \mathbb{R}$ is a periodic function of period 1 and $\tau(0) = 0$. Then, define the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ denote the group of translations in the x and y axes, clearly properly discontinuous. We take the null geodesic starting in $\gamma(0) = (0, 0)$ with velocity $v = (0, 1)$. This geodesic is defined as a parametrization of y-axis for $\mathbb{R}^2$ and it is its projection in $\mathbb{T}^2$. It is clearly null because $||v|| = 0$. We study completeness of that geodesic. To study this, we need the geodesic equations in order to understand the behavior of γ. γ is constant in the x-axis, we just need to study y direction. Recall that $\tau(0) = 0$ and that $\tau'(0)$ is a constant along all the geodesic.

$$\begin{aligned} \Gamma_{11}^2 &= 0 \\ \Gamma_{12}^2 &= 0 \\ \Gamma_{22}^2 &= \frac{\tau'(0)}{2} \end{aligned} \tag{3.1}$$

Equation (3.1) shows the results on the Christoffel symbols that are needed to study γ. There are two cases:

- If $\tau'(0) = 0$, the geodesic is periodic and its speed is constant, because $y'' = 0$. Then, γ is a complete geodesic.
- If $\tau'(0) \neq 0$, the geodesic is not complete. Taking the equation of geodesics $y'' = \frac{\tau'(0)}{2}(y')^2$, γ is constantly varying, it is either constantly growing (if $\tau'(0) > 0$) or constantly decreasing (if $\tau'(0) < 0$).

If $\tau'(0) = \varepsilon > 0$, there exist close metrics such that $\tau(0) > 0$ as well, and therefore $\mathbb{T}^2$ with these metrics is incomplete as well.

Example 3.5 is another example of a compact Lorentzian manifold that is incomplete, as we saw in example 3.1, the Clifton-Pohl torus, in the beginning of this chapter. It will be useful as well in the next section, to see an example of an complete manifold in which completeness fails to be stable.

The concept of imprisonment of a geodesic is not only essential for the study of stability of incompleteness, but also for stability of completeness.

Definition 3.6. *Let (M, g) be a differentiable manifold provided with a metric and $\gamma : M \to [0, b)$ an inextendible geodesic in the forward direction. We say that γ is* partially imprisoned *in a compact set K if there exists a sequence $\{t_n\}_n$ in $[0, b)$ such that $\lim_{n\to\infty} t_n = b$ and $\gamma(t_n) \in K$ for $n \in \mathbb{N}$. Now, suppose that $\gamma : (a, b) \to M$ is an inextendible geodesic. We say that γ is* imprisoned *in a compact set K when $Im(\gamma) \subset K$.*

Note that imprisonment implies partial imprisonment and that the whole geodesic is contained in a compact set, while partial imprisonment implies that a geodesic has a sequence of points in K, a compact set. Then, the geodesic may leave the compact set and return to it an infinite number of times. Nonimprisonment of geodesics will be a sufficient condition for the stability of incompleteness, as well as for stability of completeness.

Next theorem gives sufficient conditions for stability of incompleteness. This theorem is shown in [2, Chapter 7]. In this book, the following lemmas are essential for the proof of theorem 3.9. Another key tool we will use is an arbitrary distance d in the manifold.

Lemma 3.7. *Let (M, g) be a semi-Riemannian manifold and $\gamma : (a, b) \to M$ a geodesic of g. Let W be a neighborhood of $\gamma([t_1, t_2])$ for $t_1 < t_2 \in (a, b)$. There is a neighborhood of $\gamma'(t_1)$ in TM and a constant $\varepsilon > 0$ such that if $|g - \bar{g}|_1 < \varepsilon$ in W and if $\bar{\gamma}$ is a geodesic of $\bar{g}$ with $\bar{\gamma}'(t_1) \in V$, then the domain of $\bar{\gamma}$ includes t_2 and $\bar{\gamma}(t) \in W$ for $t_1 \le t \le t_2$.*

Lemma 3.8. *Let (M, g) be a given semi-Riemannian manifold, $\gamma : (a, b) \to M$ a geodesic of g and W be a neighborhood of $\gamma([t_1, t_2])$ for $t_1 < t_2 \in (a, b)$. If V_1 is a neighborhood of $\gamma'(t_1)$ in TM, there is $\varepsilon > 0$ and a neighborhood V_2 of $\gamma'(t_2)$ such that if $|g - \bar{g}|_1 < \varepsilon$ and if $\bar{\gamma}$ is a geodesic of $\bar{g}$ with $\bar{\gamma}'(t_2) \in V_2$, then $\bar{\gamma}'(t_1) \in V_1$ and $\gamma'(t) \in W$ for all $t_1 \le t \le t_2$. Furthermore, if γ is timelike or spacelike, V_2 and ε can be chosen so that each $v \in \bar{V}_2$ is timelike or spacelike. If γ is null, ε can be chosen such that each $\bar{g}$ has some null vectors in V_2.*

Theorem 3.9. *Let (M, g) be a semi-Riemannian manifold. If (M, g) has an incompete geodesic in the forward direction $\gamma : (a, b) \to M$, with $b < \infty$, that is not partially imprisoned in any compact set K when $t \to b$, then there exists a $\mathscr{C}^1$ neighborhood of metrics $U(g)$ such that each $\bar{g} \in U(g)$ has an incomplete geodesic. In addition, if (M, g) has an incomplete null (resp. timelike, spacelike) geodesic, $(M, \bar{g})$ has an incomplete null (resp. timelike, spacelike) geodesic.*

Proof. We start constructing the sequences $\{t_j\}_j$, $\{D_j\}_j$ and $\{L_j\}_j$. Choosing $t_0 \in (a, b)$, $t_1 > t_0$, $D_1 > 1$ such that $D_1 = d(\gamma(t_0), \gamma(t_1))$ and $d(\gamma(t_0), \gamma(t)) > D_1$ for $t_1 < t < b$. The existence of t_1 is given by the non partial imprisonment condition. Secondly, $L_0 = 0$ and $L_1 = 1 + \sup\{d(\gamma(t_0), \gamma(t)) : t_0 < t < t_1\}$. As an example, in figure 3.2 for a curve γ and $\gamma(t_0)$, there is $\gamma(t_1)$, D_1 and L_1. For the rest of the sequence suppose that t_i, D_i and L_i are given up to $j-1$. Then, $D_j > L_{j-1}$ and must fulfill $D_j = d(\gamma(t_0), \gamma(t_j))$ and $D_j < d(\gamma(t_0), \gamma(t))$ for $t_j < t < b$. The existence of D_j is given by nonimprisonment once again. Finally $L_j = 1 + \sup\{d(\gamma(t_0), \gamma(t)) : t_0 < t < t_j\}$. Note that by definition the three sequences are strictly increasing and that $\lim_{j\to\infty} D_j = \lim_{j\to\infty} L_j = \infty$.

With these sequences, we will construct a sequence of subsets of M $\{W_j\}_j$. $W_1 = \{p \in M : d(p, \gamma(t_0)) < L_1\}$, $W_2 = \{p \in M : d(p, \gamma(t_0)) < L_2\}$ and $W_j = \{p \in M : L_{j-2} < d(p, \gamma(t_0)) <$

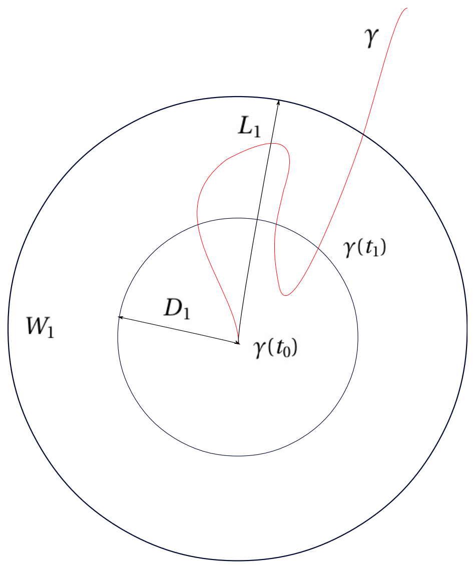


Figure 3.2: L_1, D_1 and the set W_1 for an election of a geodesic γ and $\gamma(t_0)$.

$L_j\}$ for $j > 2$. Now, we will construct another sequence $\{V_j\}_j$ of subsets of TM. We take a neighborhood of non trivial vectors of $\gamma'(t_0)$ as V_0.

- If $\gamma'(t_0)$ is timelike, we can take $\bar{V}_0$ as a set of timelike vectors.
- If $\gamma'(t_0)$ is spacelike, we can take $\bar{V}_0$ as a set of spacelike vectors.
- If $\gamma'(t_0)$ is null, we can choose V_0 with some null vectors such that $\bar{V}_0$ does not contain the zero vector.

Supposing V_i is defined up to $j-1$, we may define V_j. With this purpose, V_j is defined using lemma 3.8, that is, we obtain V_j a neighborhood of $\gamma'(t_j)$ and $\varepsilon_j > 0$ such that if $|g-\bar{g}|_1 < \varepsilon_j$ in W_j and $\bar{\gamma}$ is a geodesic of $\bar{g}$ with $\bar{\gamma}'(t_j) \in V_j$, then $\bar{\gamma}'(t_{j-1}) \in V_{j-1}$ and $\bar{\gamma}(t) \in W_j$ for all $t_{j-1} \leq t \leq t_j$. Using again lemma 3.8, we have that if γ is timelike (resp. spacelike), so is $\bar{\gamma}$. If γ is null, we assume that V_j has null vectors for $\bar{g}$. Now, lemma 3.7 implies that if $\bar{\gamma}$ is a geodesic of $\bar{g}$ such that $\bar{\gamma}'(t_j) \in V_j$, then the domain contains t_{j+1}. We may assume that each V_j of the sequence has less than $\frac{1}{2}$ diameter, that is $diam(\pi(V_j)) < \frac{1}{2}$. We can assume as well that $\varepsilon_{j+1} < \varepsilon_j$.

The points of M are in a finite number of W_j, concretely, in at most two W_j, which leads to the possibility of the construction of a continuous function $\varepsilon : M \to (0,\infty)$ such that $\varepsilon(p) < \varepsilon_j$ for $p \in W_j$.

Let $\bar{g} \in U(g) = \{\bar{g} \in \text{Pseudo}(M) : |\bar{g}-g|_1 < \varepsilon(p)\}$. We will construct $\{\bar{\gamma}_j\}_j$ a sequence of geodesics of $\bar{g}$ that contains a convergent subsequence. First, we will assume that γ is timelike or, respectively, spacelike. Secondly, we will assume that γ is null. In the first case, we define $\bar{\gamma}_j$ satisfying $\bar{\gamma}'_j(t_j) = \gamma'(t_j)$, therefore $\bar{\gamma}$ is timelike (resp. spacelike). In the second case, we choose $\bar{\gamma}_j$ such that $\bar{\gamma}'_j(t_j) \in V_j$ is null. Using the construction of V_k by lemma 3.8, we get $\bar{\gamma}'_j(t_k) \in V_k$ for $k \leq j$. For this reason, taking $\{\bar{\gamma}_j(t_0)\}_j \subset V_0$ and as V_0 has compact closure, one gets that $\{\bar{\gamma}_m(t_0)\}_m \subseteq \{\bar{\gamma}_j(t_0)\}_j$ is a convergent subsequence in $\bar{V}_0$. Using again lemma 3.8, if γ is timelike (resp. null or spacelike), $\bar{\gamma}$ is timelike (resp. null or spacelike) for $\bar{g}$.

Let $\bar{\gamma}$ be an inextendible geodesic of $\bar{g}$ starting at $\gamma(t_0)$ and with direction $\lim_{m\to\infty} \bar{\gamma}'_m(t_0) = v$. Note that each $\bar{\gamma}_m(t_j) \in \pi(V_j)$ for $j \leq m$. Besides, $\bar{\gamma}_m(t)$ converges to $\bar{\gamma}(t)$ and $\bar{\gamma}'_m(t)$ to $\bar{\gamma}'(t)$ as $m \to \infty$ for every t such that $\bar{\gamma}(t)$ is defined. To prove that $[t_0, b)$ is in the domain of $\bar{\gamma}$, we use lemma 3.7; if $\bar{\gamma}'(t_j) \in V_j$, $\bar{\gamma}(t_{j+1})$ exists. In fact, as $\bar{\gamma}_m(t_{j+1}) \in V_{j+1}$ for $m > j+1$, $\bar{\gamma}'(t_{j+1}) \in V_{j+1}$. Hence $[t_0, b)$ is in the domain of $\bar{\gamma}$. Using that $\lim_{j\to\infty} d(\gamma(t_0), \bar{\gamma}(t_j)) = \infty$, $\bar{\gamma}$ is not partially imprisoned in any compact set if $t \to b$. In addition, b is not in the domain of $\bar{\gamma}$, because if it was, the set $\bar{\gamma}([t_0, b])$ would be compact, in contradiction with $\lim_{j\to\infty} d(\gamma(t_0), \bar{\gamma}(t_j)) = \infty$. □

One aspect of interest of this proof is that an incomplete geodesic in a close metric to g can be defined using the same domain. Next corollary applies for strongly causal spacetimes, and as we saw in the causal ladder, to stably causal, causally continuous, causally simple and globally hyperbolic spacetimes.

Corollary 3.10. *Let (M, g) be a causally geodesically incomplete strongly causal spacetime. There is a neighborhood of metrics such that each $\bar{g} \in U(g)$ is geodesically incomplete.*

Proof. Suppose that γ is a causally incomplete geodesic of g. Then γ is not imprisoned nor partially imprisoned because of the strong causality condition. Thus, we achieve $U(g)$ by using theorem 3.9. □

3.4 Stability of completeness

Example 3.5 shows a case in which completeness is instable. Recall that in this example when $\tau(0) = 0$ and $\tau'(0) \neq 0$, the geodesic with direction $v = (0, 1)$ is incomplete, whereas if $\tau'(0) = 0$, this geodesic is complete. In this case, instability of the completeness of $(\mathbb{T}^2, g)$ follows from the fact that a small change in $\tau'(0)$ leads to the incompleteness of $\mathbb{T}^2$.

For the stability of completeness two conditions are required, nonimprisonment of null (resp. nonspacelike, nontimelike) geodesics and a null (resp. nonspacelike, nontimelike) pseudoconvex geodesic system. The first requirement was studied in the previous section. A pseudoconvex geodesic system is a generalization of convex hulls in $\mathbb{R}^n$. A convex hull of a compact set $K \subset \mathbb{R}^n$ is the union of all the Euclidean straight segments joining, $p, q \in K$ for all $p, q \in K$. The convex hull in $\mathbb{R}^n$ is always a compact set.

Definition 3.11. *Let (M, g) be a Lorentzian manifold, we say that (M, g) has a pseudoconvex null (resp. nonspacelike, nontimelike) geodesic system if for each compact subset $K \subset M$, all null(resp. nonspacelike, nontimelike) geodesic segments joining p, q for every $p, q \in K$ lie on a compact set $\bar{K}$.*

Then, we are in the conditions to enunciate the theorem of stability of completeness. This theorem was proven by Beem and Erlich in [1]. For the proof of this theorem, some lemmas taken from [1] will be necessary. Besides, we will need to define an auxiliar Riemannian metric h and a collection of sets $\{A(n)\}_{n\in\mathbb{N}}$. Choosing $p \in M$, $A(0) = \{p\}$ and $A(1) = \{q \in M : d(p, q) \leq 2\}$ where d is a Riemannian distance that comes from h. For the rest of $A(n)$, suppose that for any null (resp. nonspacelike or nontimelike) geodesic γ with endpoints in

$A(n-1)$ lies in $A(n)$ and additionally the condition $d(q, M\setminus A(n)) > 2$ for all $q \in A(n-1)$. The first step of the construction of these sets in in figure 3.3, where in red we have the compact set that contains all the geodesic segments whose endpoints lie in $A(1)$ and $A(2)$ is given by the recursive definition.

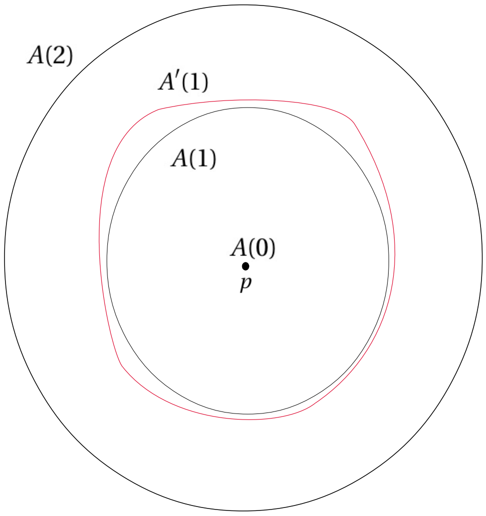


Figure 3.3: Three sets of the sequence $\{A(n)\}_n$, $A(0)$, $A(1)$ and $A(2)$. In red, we represent the "convex hull" of $A(1)$.

Lemma 3.12. *Let (M, g) be a semi-Riemannian manifold with no imprisoned null, nonspacelike or nontimelike geodesics and a null (resp. nonspacelike or nontimelike) psuedoconvex geodesic system. There exists a $\mathscr{C}^1$ neighborhood of g $U(g)$ such that if $\bar{g} \in U(g)$ has a null (resp. nonspacelike or nontimelike) geodesic $\gamma : [a, b] \to M$ with $\gamma(a), \gamma(b) \in A(n)$, then $\gamma(t) \notin A(n+4)\setminus A(n+3)$ for all $t \in [a, b]$. Furthermore, each $\bar{g}$ has no imprisoned null (resp. nonspacelike or nontimelike) geodesics and has a null (resp. nonspacelike or nontimelike) pseudoconvex geodesic system.*

Lemma 3.13. *Let (M, g) be a semi-Riemannian manifold with no imprisoned null nonspacelike or nontimelike geodesics and a null (resp. nonspacelike or nontimelike) psuedoconvex geodesic system. Let $n \in \mathbb{N}$ and $L > 0$ be fixed. There exists an integer $k > n+5$ such that if $\gamma : \mathbb{R} \to M$ is any null (resp. nonspacelike or nontimelike)geodesic of (M, g) with $\gamma(0) \in \overline{A(n)\setminus A(n-1)}$, $h(\gamma'(0), \gamma'(0)) \le L^2$ and $\gamma(t) \in \overline{A(k)\setminus A(k-1)}$, then $|t| > 2$.*

Lemma 3.14. *Let (M, g) be a semi-Riemannian manifold with no imprisoned null, nonspacelike or nontimelike geodesics and a null (resp. nonspacelike or nontimelike) psuedoconvex geodesic system. Let $n > 5$ and $L > 0$ be given. There exist $k > n+5$ and $\varepsilon > 0$ such that if $\bar{g} \in U(g)$ satisfies $|\bar{g}-g|_1 < \varepsilon$ on $A(k)\setminus A(n-5)$, then for any inextendible null (resp. nonspacelike or nontimelike)geodesic $\gamma : (a, b) \to M$ with $\gamma((a, 0]) \cap A(n-5) \ne \emptyset$, $\gamma(0) \in \overline{A(n)\setminus A(n-1)}$, $h(\gamma'(0), \gamma'(0)) \le L^2$, $\gamma(t_0) \in A(k)\setminus A(k-1)$ and $t_0 \in A(k)\setminus A(k-1)$, then $t_0 > 1$.*

Lemma 3.15. *Let (M, g) be a semi-Riemannian manifold with no imprisoned null, nonspacelike or nontimelike geodesics and a null (resp. nonspacelike or nontimelike) psuedoconvex geodesic system. Let $n, k \in \mathbb{N}$ and L be given such that $L > 0$ and $k > n+5$. There exists*

$N > 0$ such that if $\bar{\gamma}$ is a null (resp. nonspacelike or nontimelike)geodesic with $\gamma(0) \in A(n)$ and $h(\bar{\gamma}'(0), \bar{\gamma}'(0)) \le L^2$, then $h(\bar{\gamma}'(t), \bar{\gamma}'(t)) \le N^2$ for all t with $\bar{\gamma}(t) \in \overline{A(k+5)}$.

Lemma 3.16. *Let (M, g) be a semi-Riemannian manifold with no imprisoned null, nonspacelike or nontimelike geodesics and a null (resp. nonspacelike or nontimelike) psuedoconvex geodesic system. Let $n, k \in \mathbb{N}$ and L be given such that $L > 0$ and $k > n+5$. There exist $N, \varepsilon > 0$ such that if $\bar{g} \in U(g)$ with $|\bar{g} - g|_1 < \varepsilon$ on $A(k+5) \setminus A(n-5)$ and if $\gamma : (a, b) \to M$ and if $\bar{\gamma} : (a, b) \to M$ is a null (resp. nonspacelike or nontimelike)geodesic of $\bar{g}$ with $\bar{\gamma}((a, 0]) \cap A(n-5) \neq \emptyset$, $\bar{\gamma}(0) \in A(n) \setminus A(n-1)$, $h(\bar{\gamma}'(0), \bar{\gamma}'(0)) \le L^2$, $\bar{\gamma}(t_0) \in \overline{A(k) \setminus A(k-1)}$ adn $t_0 > 0$, then $h(\bar{\gamma}'(t_0), \bar{\gamma}'(t_0)) \le N^2 + 1$*

Using these five lemmas we can prove the following theorem of stability of completeness.

Theorem 3.17. *Let (M, g) be a semi-Riemannian manifold null (resp. nonspacelike or nontimelike) geodesically complete, provided with a null (resp. nonspacelike or nontimelike) pseudoconvex geodesic system and non imprisoned in a compact set null (resp. nonspacelike or nontimelike) geodesics. Then, there exists a $\mathscr{C}^1$ neighborhood $U(g)$ such that $(M, \bar{g})$ is null (resp. nonspacelike or nontimelike) geodesically complete and besides, it has a null (resp. nonspacelike or nontimelike) pseudoconvex geodesic system and non imprisoned null (resp. nonspacelike or nontimelike) geodesics.*

Proof. Using lemma 3.12, we find a $\mathscr{C}^1$ neighborhood of metrics $U_1(g)$ fulfilling the conditions of this lemma, which is given by a positive funciton $\delta : M \to (0, \infty)$. Now we define four sequences $\{n_j\}_j$, $\{k_j\}_j$, $\{\varepsilon_j\}_j$ and $\{L_j\}_j$ for $j = 1 \in \mathbb{N}$. Initially, $n_1 = 6$ and $L_1 = 1$. k_1 is calculated using lemma 3.13 and ε_1 with lemma 3.16. For the rest of the sequence, suppose that n_j and L_j are given, and that k_j is obtained by using lemma 3.13. Then, $n_{j+1} = k_j$ and using the lemma 3.16 we obtain the constants ε and N, we set $\varepsilon_j = \varepsilon$ and $L_{j+1} = \sqrt{N^2 + 1}$.

We can construct a continuous function $\varepsilon : M \to (0, \infty)$ with the conditions:

- $\varepsilon(p) < \delta(p)$ for $p \in M$.
- $\varepsilon(p) < \varepsilon_j$ for $p \in A(k_j + 5) \setminus A(n_j - 5)$ and $j = 1 \in \mathbb{N}$

Note that this function is well defined because each $p \in M$ is at most in two sets of the form $A(k_j + 5) \setminus A(n_j - 5)$. With this function we construct a $\mathscr{C}^1$ neighborhood of metrics $U(g) = \{\bar{g} \in U_1(g) : |\bar{g} - g|_1 < \varepsilon\}$. As $U(g) \subset U_1(g)$, every $\bar{g} \in U(g)$ has a pseudoconvex null (resp. nonspacelike or nontimelike)geodesic system and has no imprisoned null (resp. nonspacelike or nontimelike)geodesics, but completeness has to be proven.

Let $\bar{\gamma} : (a, b) \to M$ be any inextendible null (resp. nonspacelike or nontimelike)geodesic in $(M, \bar{g})$, with $\bar{g} \in U(g)$. As the sequence $\{A(n)\}_n$ covers M, there is a first j_0 such that $Im(\bar{\gamma}) \cap (A(n_{j_0}) \setminus A(n_{j_0} - 1)) \neq \emptyset$. We assume that $\bar{\gamma}(0) \in A(n_{j_0+1}) \setminus A(n_{j_0+1} - 1)$, $h(\bar{\gamma}'(0), \bar{\gamma}'(0)) \le L^2_{j_0+1}$ and that $Im(\bar{\gamma}) \cap (A(n_{j_0}) \setminus A(n_{j_0} - 1))$ for negative values of t. Applying lemmas 3.14 and 3.16 to n_j and k_j with $n_{j+1} = k_j$, for each $j \ge j_0$, $\bar{\gamma}$ increases t by, at least, one unit each time it goes from $(A(n_j) \setminus A(n_j - 1))$ to $(A(n_{j+1}) \setminus A(n_{j+1} - 1))$. Thus, $b = \infty$, that is, $\bar{\gamma}$ is complete in the forward direction. In the reverse direction and proceeding in the same way, $a = -\infty$ and hence, $\bar{\gamma}$ is complete.

□

If a manifold is null, nonspacelike and nontimelike complete and fulfills the conditions of the theorem for these three kinds of geodesics, there exists a neighborhood of metrics $U(g)$ such that $(M, \bar{g})$ is null, nonspacelike and nontimelike complete. Note that in this case, we can take the neighborhoods of metrics $U_1(g)$, $U_2(g)$, $U_3(g)$ ensuring null (resp. nonspacelike or nontimelike) completeness, and define $U(g) = U_1(g) \cap U_2(g) \cap U_3(g)$. From this point, if the type of completeness is not specified, it will be assumed that the manifold is complete in the three senses.

One trivial example of a stable complete Lorentzian manifold is $\mathbb{L}^n$. As geodesics are straight lines, all its geodesics are non imprisoned. Besides, it has a pseudoconvex geodesic system. Therefore, by theorem 3.17, there exists a neighborhood of metrics $U(\eta)$ such that if $\bar{g} \in U(g)$, $(\mathbb{R}^n, \bar{g})$ is a complete manifold.

If a spacetime (M, g) is globally hyperbolic and nonspacelike complete, we can ensure that there is a $\mathscr{C}^1$ neighborhood of metrics in which M is complete.

Theorem 3.18. *Let (M, g) be a globally hyperbolic spacetime. If M is nonspacelike complete, there is a $\mathscr{C}^1$ neighborhood of metrics $U(g)$ such that each $\bar{g}$ is nonspacelike complete.*

Proof. In a globally hyperbolic spacetime, non imprisonment is ensured. In addition, it has a pseudoconvex nonspacelike geodesic system. It is enough to prove this so that we are in the conditions of theorem 3.17. Let K be a compact set in M. For each $p \in K$, let $q \in I^-(p)$ and $r \in I^+(p)$. The open set $U(p) = I^+(q) \cap I^-(r)$ contains p and the compact set K can be covered with a finite number of this kind of sets. As the closure of each $U(p_i)$ is $J^+(q_i) \cap J^-(r_i)$, it is compact and we can ensure that K is contained in a compact set $\tilde{K} = U(\bar{p}_1) \cup \ldots \cup U(\bar{p}_n)$ such that the nonspacelike geodesics joining any two points of K are contained in $\tilde{K}$. □

Chapter 4

Completeness in manifolds close to $\mathbb{L}^n$

In this chapter, we study completeness and stability of Lorentzian manifolds that are isometric to $\mathbb{L}^n$ out of a compact set. In spite of the existence of general available criteria for the stability of completeness, none of them seem to apply here. However, convexity arguments will allow us to resolve this problem. For this argument, positive definiteness of the second fundamental form of affine spheres will become essential. We will prove the geodesic completeness of Lorentzian manifolds that are isometric to $\mathbb{L}^n$ out of a compact set is stable, in the sense that there exists a special kind of neighborhood of metrics $U_K(g)$ such that if $\bar{g} \in U_K(g)$, then $(M, \bar{g})$ is complete. The second fundamental form will be studied in the first section, whereas in the second section, results on stability of completeness for some specific kinds of Lorentzian manifolds are presented.

4.1 Preliminaries on the second fundamental form of an affine connection

The second fundamental form is a key ingredient along this chapter. Recall that once an affine connection has been defined, each hypersurface has a second fundamental form σ^Z with respect to each choice of a transverse vector field Z. Let $X, Y \in \mathfrak{X}(S)$; given an affine connection ∇ in M, a hypersurface S and a transverse vector Z to the hypersurface S, one can write:

$$\nabla_X Y = (\nabla_X Y)^{\mathrm{T}} + (\nabla_X Y)^{\perp}. \tag{4.1}$$

In equation (4.1), $(\nabla_X Y)^{\mathrm{T}}$ denotes the tangent component of $\nabla_X Y$. Classically, $(\nabla_X Y)^{\perp}$ denotes the orthogonal component of $\nabla_X Y$ (once a Riemannian metric is provided), nevertheless, we take this notation for the transverse component of $\nabla_X Y$. The second fundamental form in S with respect to the transverse vector Z is $\sigma^Z : \mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathscr{F}(S)$, defined by the equality:

$$(\nabla_X Y)^{\perp} = \sigma^Z(X, Y) Z. \tag{4.2}$$

Next, we will consider the second fundamental form for affine spheres embedded in $\mathbb{L}^n$, so that only the affine structure of $\mathbb{R}^n$ will be essential, even though we can take into account its usual Euclidean structure, or any other auxiliary scalar product, for some choices. Fix $\{S_R\}_{R>0}$ a foliation of spheres of $\mathbb{R}^n \setminus \{0\}$ where R denotes the radius of the spheres, and N to be the unit inner transverse vector of every sphere, $N = -\frac{1}{t} P$, where $P = \sum_i X^i \frac{\partial}{\partial x^i}$ is

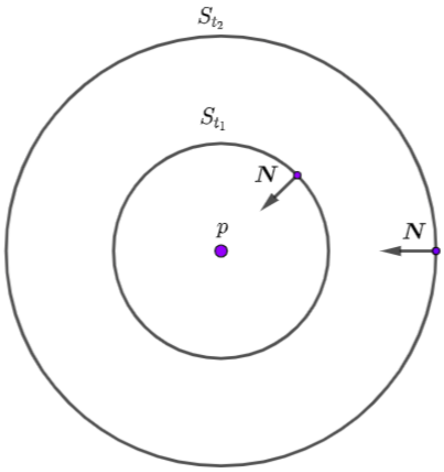


Figure 4.1: Two hypersurfaces S_R and the our choice of transverse vector in a point of each one.

the position vector. Take $S_R = \{x_1^2 + \ldots + x_n^2 = R^2\}$, where $x_1, \ldots, x_n$ denote the canonical coordinates in $\mathbb{R}^n$. A representation of these spheres and its normal vectors is in figure 4.1. We are going to see that S_R is strongly convex in $\mathbb{L}^n$. As the Levi-Civita connections are the same for both the Lorentz-Minkowski metric and the Euclidean one due to the equality in their Christoffel symbols $\Gamma_{ij}^k = 0$, $i, j, k = 1, \ldots, n$, we can work with the Euclidean metric to calculate the second fundamental form with respect to the unitary inner vector. We can fix $\sigma = \sigma^N$ and write, which:[1]

$$\nabla_X^0 Y = (\nabla_X^0 Y)^{\mathrm{T}} + (\nabla_X^0 Y)^{\perp}. \tag{4.3}$$

For our choice, $(\nabla_X^0 Y)^{\perp}$ is actually a field normal to the hypersurface. In the rest of the chapter, in the decomposition (4.1), $(\nabla_X^0 Y)^{\mathrm{T}}$ will be the tangent component, whereas $(\nabla_X^0 Y)^{\perp}$ will be the transverse component in the direction of N. The second fundamental form is defined as $\sigma(X, Y)$ in (4.2). Using that ∇^0 is the Levi-Civita connection we have:

$$(\nabla_X^0 Y)^{\perp} = -\frac{1}{R}\left\langle \nabla_X^0 Y, -\frac{1}{R}P \right\rangle P = -\frac{1}{R}\left\langle \nabla_X^0 \frac{1}{R}P, Y \right\rangle P$$

To calculate $\nabla_X^0\left(\frac{1}{R}P\right)$, recall that $\Gamma_{ij}^k = 0$ for $i, j, k = 1, \ldots, n$ and that $\frac{\partial x^i}{\partial x^j} = \delta_j^i$, thus $\nabla_X^0\left(\frac{1}{R}P\right) = \frac{1}{t}X$ and $(\nabla_X^0 Y)^{\perp} = -\left(\frac{1}{R}\right)^2 \langle X, Y\rangle P = \frac{1}{R}\langle X, Y\rangle N$. As the usual metric is positive definite, so is the second fundamental form.

Now, observe that the coefficients of the second fundamental form for the Levi-Civita connection depend on the coefficients of the metric and its derivatives. Note that N is the normal vector to S_R with the Euclidean scalar product, but working with arbitrary connections this concept is lost. As we are going to work with connections ∇ different to ∇^0, we will be interested in the computation of σ for such a nabla maintaining both the hypersurfaces S_R and the transverse vector N. Taking a coordinate system $y^1, \ldots, y^n$ in an open set $U \subset \mathbb{R}^n$ such that the first $n-1$ vectors yield coordinates of each sphere S_R and $N = -\frac{\partial}{\partial y^n}$ is

[1]Every affine space has a natural affine connection ∇^0. This fulfills the following condition: the coordinate fields are parallel, independently of the coordinates chosen. When a manifold is endowed with a metric, the Levi-Civita connection has this property.

the transverse vector (take, for example, $N = -\frac{\partial}{\partial r}$ in the spherical coordinates outside zero), one can write for any connection ∇:

$$\sigma\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = -dy^n(\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j}) = -\Gamma^k_{ij} dy^n\left(\frac{\partial}{\partial y^k}\right) = -\Gamma^n_{ij}.$$

In particular, this expression shows that the coefficients of the second fundamental form only depend on the Christoffel symbols, which in turn depend on the coefficients of the metric and its first derivatives if ∇ is the Levi-Civita connection.

Lemma 4.1. *For the Lorentzian scalar product of $\mathbb{L}^n$, there exists a $\mathscr{C}^1$ neighborhood $U(\eta)$ of Lorentzian metrics for the Lorentzian scalar product of $\mathbb{L}^n$ such that for all the metrics $g \in U(\eta)$ the second fundamental form σ of each affine sphere S_R is positive definite for all $R > 0$.*

Proof. As calculated above, the coefficients of the second fundamental form for $\frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial y^j}$ are $-\Gamma^n_{ij}$. On the other hand, we have calculated that $(\nabla^0_X Y)^\perp = \frac{1}{R}\langle X, Y\rangle N$, which leads to the equality $(\Gamma^0)^n_{ij} = -\frac{\delta_{ij}}{r^2}$ for $i, j = 1, \dots, n-1$. Thus, from the equation for Γ^n_{ij} in terms of the metric, it can be immediately followed that σ is positive definite for $\mathscr{C}^1$ close metrics. □

Recall that a compact hypersurface is said to be convex if,locally (and then globally, because of the particular structure of $\mathbb{R}^n$), the tangent plane of a point only intersects the manifold in the point. We say that the manifold is strictly convex if, the tangent plane in a point only intersects the manifold in that point. This condition implies that the second fundamental form defined for the inner vector is positive semidefinite and, moreover, positive semidefiniteness of the second fundamental form implies strict convexity. In case the second fundamental form is positive definite, we say that the manifold is strongly convex. Next lemma shows that if S_R is strongly convex, geodesics that leave a sphere of the foliation do not return to it.

Lemma 4.2. *Let $(\mathbb{R}^n, \nabla)$ a differentiable manifold provided with a connection ∇, $\{S_R\}_{R>0}$ a foliation with spheres of $\mathbb{R}^n \setminus \{0\}$, and $\{B_R\}_{R>0}$ the closed Euclidean balls enclosed by S_R for every $R > 0$. If S_R is strongly convex for every $R > 0$, once a geodesic γ leaves B_R, it never returns to it.*

Proof. Suppose that $\gamma : [a, b] \to \mathbb{R}^n$ is a segment of geodesic which leaves a ball B_{R_0} at a and returns to it at b. As the parameter R of the foliation is also the radius of the spheres, it can be regarded as a continuous function on $\mathbb{R}^n$, and $t \circ \gamma$ will take a maximum at some point in (a, b). Without loss of generality suppose it takes its maximum R_{max} at $s_{max} \in (a, b)$, which can be assumed to be $s_{max} = 0$. Then:

(a) $(R \circ \gamma)'(0) = dR(\gamma'(0)) = 0$, leading to $\gamma'(0) \in T_{\gamma(0)} S_{R_{max}}$.

(b) $(R \circ \gamma)''(0) \leq 0$.

However, calculating these derivatives explicitly, we have:

$$(R \circ \gamma)''(0) = \frac{d^2}{ds^2}(R \circ \gamma) = \frac{d}{ds}(dR(\gamma')) = (\nabla_{\gamma'} dR)(\gamma') = \text{Hess}R(\gamma', \gamma'). \tag{4.4}$$

About the last equality, recall the expression of the hessian (1.13). Applying dR to (4.2) we have $dR(\nabla_X Y) = dR((\nabla_X Y)^T)+dR((\nabla_X Y)^\perp) = -\sigma(X,Y)$, as $dR(\sigma(X,Y)N) = \sigma(X,Y)dR(N) = -\sigma(X,Y)$. Now, it is possible to find a relation between the hessian and the second fundamental form of S_R given by the vector N. Recalling that R is the radius function, for any $X \in T_pS_R$, $dR(X) = X(R) = 0$, and using the expressions for the hessian and the second fundamental form, we have:

$$0 = X(Y(R)) = X(dR(Y)) = (\nabla_X dR)Y + dR(\nabla_X Y) = \text{Hess}R(X,Y) - \sigma(X,Y) \quad \forall X,Y \in T_pS_{R_{max}}.$$

Therefore, $\text{Hess}R(X,Y) = \sigma(X,Y)$. The expression (4.4) shows that

$$(R\circ\gamma)''(0) = \text{Hess}R(\gamma',\gamma') = \sigma(\gamma',\gamma') > 0,$$

which contradicts the existence of a maximum in 0 (i.e. the condition (b) above) and thus, the fact that a geodesic can leave a B_{R_0} and return to it afterwards. □

4.2 Main Lorentzian result on completeness

In this section, we are going to work mainly with Lorentzian manifolds that are isometric to $\mathbb{L}^n$ out of a compact subset, or close to them. In what follows, $K \subset M$ will be a compact connected n-submanifold with boundary ∂K. That is, K is the closure of a non-empty open subset $\mathring{K}$ whose boundary is a smooth hypersurface embedded in M. The following definition gives the basic type of manifold that will be studied along this chapter.

Definition 4.3. *A Lorentzian manifold M has a* Lorentz-Minkowski end *if $M \setminus K$ is isometric to $\mathbb{L}^n \setminus K_0$, where $K \subset M$ is a compact connected set with smooth connected boundary that is the closure of a non-empty open set and $K_0 \subset \mathbb{L}^n$ is a compact set of $\mathbb{L}^n$. In particular, K is a* compact ball *in M when its boundary ∂K is equal to an Euclidean sphere S_R (see section 4.1) for some $R > 0$ in $\mathbb{L}^n$. In this case, we say that M has a* spherical Lorentz-Minkowski end.

We will fix the isometry ϕ of the definition along this chapter so that we will identify points of $M \setminus \mathring{K}$ to $\mathbb{L}^n \setminus \mathring{K}_0$. As K is the closure of an open subset, the natural notion of geodesic completeness will take into account the geodesics ending in the boundary. Then, it is possible to find geodesics that cannot be defined for all $s \in \mathbb{R}$. The concept of geodesic completeness in a manifold with boundary is not exactly the same as in a smooth manifold without boundary, because geodesic ending at the boundary will be regarded as complete even if they are not defined for all $s \in \mathbb{R}$.

Definition 4.4. *Let K be a smooth manifold with boundary. K is* geodesically complete *if every inextendible geodesic γ in K is defined on a closed interval I. That is, if I is lower or upper bounded, γ has an endpoint in K.*

It is clear that when a manifold M is geodesically complete, any n-submanifold N with boundary, closed as a subset, is complete as well, because geodesics are either defined on $\mathbb{R}$ or have an endpoint.

In chapter 3, the concept of $\mathscr{C}^1$ neighborhood of a metric was defined. In this chapter however, the hypotheses of the theorem 3.17 of stability of completeness may not hold, and it is necessary to find new conditions to ensure stability. It is for this reason that neighborhoods of metrics up to K are introduced.

Definition 4.5. *Let (M,g) be a Lorentzian manifold. A set of metrics $U_K(g) \subset Lor(M)$ is a* neighborhood of g up to K *if there exists some continuous function $\delta : M \to [0,\infty)$ such that $U_K(g) = \{\bar{g} \in Lor(M) : |g-\bar{g}|_1 \leq \delta \, in \, K, |g-\bar{g}|_1 < \delta$ in $M \setminus K\}$ where $\delta(p) = 0$ if and only if $p \in K$.*

Notice that, when all the metrics are restricted to $M \setminus K$, all those in $U_K(g)$ constitute a usual $\mathscr{C}^1$ neighborhood of g.

One of the features that will be studied in this section is the holonomy group of K, as well as its implications on the completeness of M. In particular, we will study precompact holonomies.

Observation 4.6. It makes sense to extend the notion of holonomy group to any connected manifold K with boundary, allowing the curves to lie on the boundary. When the holonomy group is calculated in an interior point, the curves that touch the boundary can be approximated to curves in $\mathring{K}$. Considering continuity of parallel transport, both holonomy groups of K and $\mathring{K}$ will have the same closure.

In the next proposition, we study completeness in the particular case that K is a compact ball, that is, $K_0 = B_{R_0}$, where B_{R_0} is the ball of radius R_0.

Proposition 4.7. *Let (M,g) be a Lorentzian manifold such that M has an spherical Lorentz-Minkowski end and $M \setminus \mathring{K}$ is isometric to $\mathbb{L}^n \setminus \mathring{B}_{R_0}$, where B_{R_0} is the ball of radius t_0. If K is geodesically complete, then M is complete. In particular, if K has precompact holonomy, M is complete.*

Proof. Let $\gamma : [0,b) \to M$ be an inextendible geodesic in M. If $Im(\gamma) \subset M \setminus \mathring{K}$, γ can be treated as a geodesic in $\mathbb{L}^n$, therefore, γ is a complete geodesic; indeed, even if γ intersects ∂K, because of continuity of the Levi-Civita connection, γ is again treated as a geodesic in $\mathbb{L}^n$ and therefore complete. Secondly, if $Im(\gamma) \cap \mathring{K} \neq \emptyset$, there are two possibilities. If γ enters into K and it does not leave again, γ is complete in the forward direction due to the completeness hypothesis in K. Otherwise, γ leaves K at least once. However, in this case, it will not return to it, because when it leaves it can be treated as geodesic in $\mathbb{L}^n$ leaving a convex set.

Precompact holonomy in the compact set K implies that every geodesic in K is extendible in K, as shown in theorem 2.7. Thus, completeness of M follows. □

As a remark of the previous proposition, the hypothesis of K being a compact ball has been essential. This proof would not hold if the compact set K would have been a general compact set. Next theorem is a result on stability of completeness in manifolds that are $\mathbb{L}^n$ out of a compact ball, as the ones of the previous proposition.

Theorem 4.8. *Under the hypotheses of proposition 4.7, there exists a neighborhood of Lorentzian metrics up to K $U_K(g)$ of g such that $\bar{g} \in U_K(g)$ $(M,\bar{g})$ is complete.*

Proof. Our manifold will be divided in two components, $M \setminus \mathring{K}$ and K. Studying $M \setminus \mathring{K}$ is equivalent to study $\mathbb{L}^n \setminus \mathring{B}_{R_0}$, which is complete as a manifold with boundary. Then, notice that there exists a $\mathscr{C}^1$ neighborhood $U_1'(\eta)$ such that if $\bar{g} \in U_1'(\eta)$, $(\mathbb{L}^n, \bar{g})$ is complete as $\mathbb{L}^n$ fulfills the conditions of theorem 3.17.

Completeness in $\mathbb{L}^n$, implies completeness in $\mathbb{L}^n \setminus \mathring{B}_{R_0}$, and therefore, completeness in $M \setminus \mathring{K}$. Then, there exists a neighborhood of complete metrics for $M \setminus \mathring{K}$:

$$U_1(g) = \{\bar{g} \in \mathrm{Lor}(M \setminus \mathring{K}) : |g - \bar{g}|_1 < \delta_1\}. \tag{4.5}$$

On the other hand, recall that for ensuring K to be complete, even if the metric of K is g, it is precise that no new geodesics enter into it. In $\mathbb{L}^n$, the strong convexity of $\{S_R\}_R$ prevents that geodesics leaving $\mathring{B}_{R_0}$ return to it; so, in M the strong convexity of hypersurfaces isometric to $\{S_R\}_R$ prevents that geodesics leaving K can come back, as in lemma 4.2. As the coefficients of the second fundamental form of $\{S_R\}_R$ depend on the coefficients of g and its first derivatives, it is possible to find a $\mathscr{C}^1$ neighborhood of metrics $U_2(\eta)$ in $\mathbb{L}^n \setminus \mathring{B}_{R_0}$, as we could see in lemma 4.1 in which the strong convexity of the collection $\{S_R\}_R$ is preserved. The $\mathscr{C}^1$ neighborhood for g in $M \setminus \mathring{K}$ can be written as:

$$U_2(g) = \{\bar{g} \in \mathrm{Lor}(M \setminus \mathring{K}) : |g - \bar{g}|_1 < \delta_2\}. \tag{4.6}$$

Then, if $\bar{g} \in U_1(g) \cap U_2(g)$, $(M \setminus \mathring{K}, \bar{g})$ is a complete manifold with boundary and the collection $\{S_R\}_R$ is strongly convex for each $\bar{g}$. Having these two neighborhoods, we can build a neighborhood up to K and check that M is complete with any of these metrics. Indeed, let t be the radius function and δ be the function defined as follows:

$$\delta(p) = \begin{cases} 0 & p \in K \\ e^{-\frac{1}{R(p)-R_0}} \min(\delta_1(p), \delta_2(p)) & p \notin K \end{cases} \tag{4.7}$$

This function is continuous and equal to zero in K. Finally, the neighborhood up to K can be given as:

$$U_K(g) = \{\bar{g} \in \mathrm{Lor}(M) : |g - \bar{g}|_1 \le \delta, |g - \bar{g}|_1 < \delta \text{ in } M \setminus K\} \tag{4.8}$$

It is enough to check completeness of geodesics in the forward direction, that is, $\gamma : [0, b) \to M$, because the reparameterization $\gamma(-t)$ is a geodesic as well. Let $\gamma : [0, b) \to M$ be an inextendible geodesic in $(M, \bar{g})$ for $\bar{g} \in U_K(g)$.

- Suppose that γ is imprisoned in K. As the metric in it remains unchanged, γ in K is one of the geodesics of (M, g), therefore, it is complete in the forward direction.

- If $Im(\gamma) \cap K \neq \emptyset$ and γ points outwards K at some point $\gamma(s_0)$, by lemma 4.2, it does not return to it, because the collection $\{S_R\}_{R \ge R_0}$ is strongly convex for every metric $\tilde{g}$ in $U_K(g)$, as they belong to $U_2(g)$. Then, γ remains in $M \setminus \mathring{K}$ leading to its completeness in the forward direction, which holds for any metric $\bar{g}$ in $U_1(g)$ above, and thus, in $U_K(g)$.

- If $Im(\gamma) \cap K = \emptyset$, it is clear that it is complete, because it can be studied out of K.

□

Observation 4.9. Moreover, we can find an actual $\mathscr{C}^1$ neighborhood of g for some especial cases. We can prove that (M, g) has a pseudoconvex geodesic system. Let $A \subseteq M$ be a compact set, and let $B \subseteq M$ be a closed ball that encloses $A \cup K$. Then, γ joining $p, q \in A$ cannot leave B, because ∂B can be regarded as a sphere in $\mathbb{L}^n$, and by lemma 4.2, once a geodesic leaves a sphere in $\mathbb{L}^n$, it cannot return to it.

Thus, assuming that (M, g) has no imprisoned geodesics in K, the hypotheses of theorem 3.17 hold and a the neighborhood $U(g)$ such that if $\bar{g} \in U(g)$, $(M, \bar{g})$ is complete is found directly.

In the next two sections, we will see generalizations of this result. First, we will see that if M is $\mathbb{L}^n$ up to a compact connected set with connected boundary, we can ensure its completeness when K has precompact holonomy. Furthermore, there exists a $\mathscr{C}^1$ neighborhood up to a compact ball enclosing K such that M is complete. Secondly, we will see that a manifold can have two or more Lorentz-Minkowski ends, see definition 4.15. In this case, if the ends are spherical, completeness of M in ensured by completeness of K, in fact, completeness is stable in this case. In this section, it can also be seen a generalization in case the boundary of K in each end is an arbitrary connected hypersurface instead of an sphere.

4.2.1 M is $\mathbb{L}^n$ out of a connected compact manifold with connected boundary

Up to this point, if M has a spherical Lorentz-Minkowski end and M is complete, the existence of a special neighborhood of metrics in which M is complete has been proven. Nevertheless, we cannot ensure the existence of neighborhoods up to K when K is an arbitrary manifold with boundary. When K is a compact manifold with boundary, ∂K is a finite union of disjoint connected components $\partial_i K$ $i = 1, \ldots, n$, which are differential hypersurfaces in M. Now, recall the celebrated Jordan-Brouwer theorem.

Theorem 4.10. *Let N be a compact connected hypersurface inside $\mathbb{R}^n$. Then N divides $\mathbb{R}^n$ into two connected components, one bounded N_1, called the inside of N and other one N_2 that is unbounded. In fact, $\bar{N}_1$ is a smooth manifold with boundary.*

By this theorem, each $\partial_i K$ divides M into two connected components. It can be immediately deduced from theorem 4.10 because M is $\mathbb{L}^n$ out of K. A strong result in completeness as the one given in the previous section is not achieved for a general K. Nevertheless, if K is a compact connected manifold with connected boundary and if we add the condition of precompact holonomy, we can achieve a result in completeness. In order to obtain this improvement, we will use Lipschitz curves.

Definition 4.11. *Let (M, g) be a Riemannian manifold and d the distance induced by g. We say that $\gamma : I \to M$ is Lipschitz if there exists a constant $C > 0$ such that*

$$d(\gamma(t'), \gamma(t)) \leq C|t - t'| \tag{4.9}$$

for every $t, t' \in I$. We say that $\gamma : I \to M$ is locally Lipschitz if for every $t \in I$ there exists an interval I' with $t \in I'$ such that $d(\gamma(t'), \gamma(t)) \leq C|t - t'|$ for $t' \in I'$ and for some C depending on t.

Observation 4.12. Note that if a curve in (M, g) is Lipschitz, so is locally Lipschitz. Moreover, it is easy to check that if a curve is locally Lipschitz for g, it is locally Lipschitz for every Riemannian metric. Moreover, if I is compact, the notions of locally Lipschitz and Lipschitz are the same.

Finally, observe that parallel transport is defined for Lipschitz curves. Indeed, general ODE theory states that the criteria of existence and uniqueness apply to Lipschitz functions, therefore, parallel transport can be extended to Lipschitz curves as well.

Next lemma is a result on the holonomy group of a compact ball $\bar{K}$ enclosing K in M with $Hol_p(\nabla^K)$ precompact.

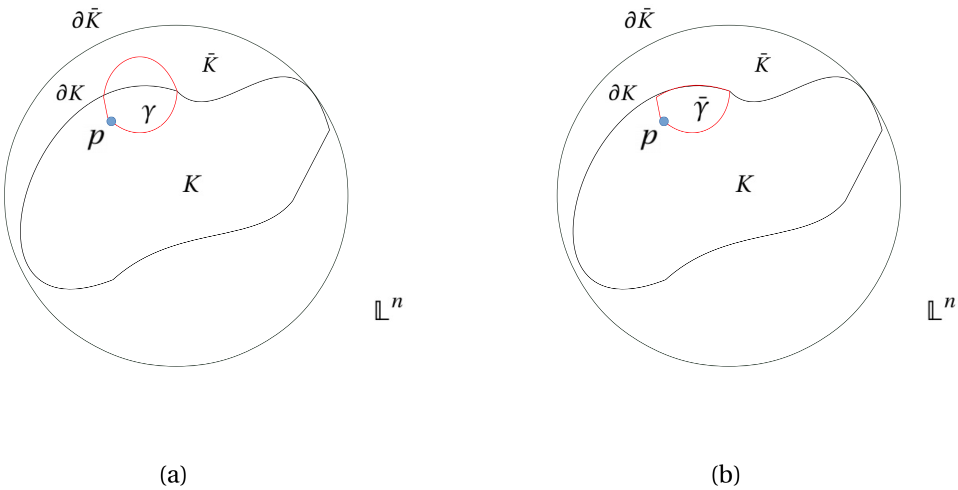


Figure 4.2: Representation of the sets K and $\bar{K}$ and two curves, one in (a) and one in (b), with the same parallel transport in $\bar{K}$.

Lemma 4.13. *Let M be $\mathbb{L}^n$ up to K and K a compact manifold with connected differentiable boundary in M and having precompact holonomy. If $\bar{K}$ is a compact ball enclosing K, then $\bar{K}$ has precompact holonomy.*

Proof. First, we show for each curve in $\bar{K}$ leaving and returning to K once, one can find a second curve contained in K that has the same parallel transport as the first above. Let $\gamma_1 : [0,b] \to \bar{K}$ be a curve that leaves K once, that is $\gamma_1(t_1), \gamma_1(t_2) \in \partial K$, $t_1 < t_2$ and $\gamma([t_1,t_2]) \subset \bar{K} \setminus K^o$. Out of K, the connection is the usual one and the parallel transport is trivial, that is, $\tau_{\gamma_1([t_1,t_2])} \equiv id_{\mathbb{R}^n}$. Any curve in ∂K will have the same parallel transport as out of K, that is if $\beta : [t_1,t_2] \to \partial K$ is a curve with $\beta(t_1) = \gamma_1(t_1)$ and $\beta(t_2) = \gamma_1(t_2)$, then $\tau_\beta = \tau_{\gamma_1([t_1,t_2])} \equiv id_{\mathbb{R}^n}$. Therefore, this equality holds:

$$\tau_{\gamma_1} = \tau_{\gamma_1[t_2,b]} \circ \tau_{\gamma_1[t_1,t_2]} \circ \tau_{\gamma_1[0,t_1]} = \tau_{\gamma_1[t_2,b]} \circ \tau_{\beta[t_1,t_2]} \circ \tau_{\gamma_1[0,t_1]}. \tag{4.10}$$

Defining the curve in K

$$\bar{\gamma}_1 = \begin{cases} \gamma_1 & 0 \le t \le t_1 \\ \beta & t_1 \le t \le t_2 \\ \gamma_1 & t_2 \le t \le b \end{cases} \tag{4.11}$$

we find that $\tau_{\gamma_1} = \tau_{\bar{\gamma}_1}$. In figure 4.2a, the loop γ leaves K once. In figure 4.2*b*, the loop $\bar{\gamma}$ has the same parallel transport as γ and is completely contained in K.

As $\bar{K}$ is connected, $Hol(\nabla^{\bar{K}})$ does not depend on the point chosen for its calculation. Let $\gamma : [a,b] \to \bar{K}$ be a piecewise differentiable loop contained in $\bar{K}$ with its endpoint p in K. As $\bar{K} \setminus K$ is open, $\gamma^{-1}(\bar{K} \setminus K)$ is open. Let $\gamma(t) \in M \setminus K$, then, there exists a maximal open interval $(a,(t),b(t))$ such that $\gamma|_{(a(t),b(t))}$ lies on $M \setminus K$. In particular, $\gamma(a(t)), \gamma(b(t)) \in \partial K$. Considering ∂K with a Riemannian metric h, for example, the metric induced by the Euclidean one in $\mathbb{R}^n$ and substitute each $\gamma|_{(a(t),b(t))}$ by a minimizing geodesic from $\gamma(a(t))$ to $\gamma(b(t))$. We define the curve $\bar{\gamma} : [a,b] \to K$ as the curve γ after subtituting the segments in which the geodesic left K.

$\bar{\gamma}$ is a Lipschitz curve. This is because in $\mathring{K}$ the geodesic was Lipschitz and so is in ∂K. As ∂K is a differentiable hypersurface, it can be seen as $F^{-1}(0)$ for a differentiable function $F : M \to \mathbb{R}$, that, in particular, is Lipschitz. From observation 4.12, the parallel transport of $\bar{\gamma}$ can be defined. Furthermore, the parallel transport of $\bar{\gamma}$ coincides with the parallel transport of γ, because in each $\bar{\gamma}|_{(a(t),b(t))}$ for $\gamma(t) \in M \setminus K$ the parallel transport is the same as for $\gamma|_{(a(t),b(t))}$, because both segments belong to $\mathbb{L}^n$ and are homotopic. □

Next theorem shows that in case K has precompact holonomy and connected boundary, M is complete and there exists a compact ball $\bar{K}$ containing K and an up to $\bar{K}$ neighborhood of complete metrics.

Theorem 4.14. *Let (M,g) be a Lorentzian manifold that has a Lorentz-Minkowski end and $K \subset M$ a compact set, which has precompact holonomy. Then, (M,g) is complete and there exists a neighborhood $U_{\bar{K}}(g)$ of complete metrics, where $\bar{K}$ is a compact ball enclosing K.*

Proof. The case in which K has precompact holonomy and connected differentiable boundary can be reduced to the case in which M is $\mathbb{L}^n$ out of a compact ball. Taking in $M \setminus K$ any hypersurface isometric to a sphere we define the compact ball $\bar{K}$ as the space enclosed by the sphere, that is, if the sphere is S_{R_0}, $\bar{K} = B_{R_0} \setminus \mathring{K} \cup K$. By lemma 4.13, $\bar{K}$ has precompact holonomy and thus, $\bar{K}$ is complete as a manifold with boundary. By the proposition 4.7, (M,g) is complete and by theorem 4.8, there exists $U_{\bar{K}}(g)$ a neighborhood of metrics up to $\bar{K}$ such that for $\bar{g} \in U_{\bar{K}}(g)$, $(M,\bar{g})$ is complete. □

We will show an example in which we see that out of the compact set we must have $\mathbb{L}^n$ and not any flat Lorentzian manifold. Let M be the Misner cylinder and $K = \{(x,y) \in M, (x-\frac{3}{2})^2 + y^2 \leq \varepsilon^2\}$ for $\varepsilon < \frac{1}{2}$. It is clear that $M \setminus \mathring{K}$ and K are two flat complete manifolds, but M is not. Even though we are in a flat manifold, the parallel transport in the boundary is not the same as in $M \setminus \mathring{K}$, because $M \setminus \mathring{K}$ is not $\mathbb{L}^2$ minus an open set and the curves are not homotopic. Take for example a curve $y = 0$ in $M \setminus \mathring{K}$ and a curve in ∂K that joins its endpoints, defined as in the proof of 4.13. The manifold is flat, but both curves are not homotopic, and the parallel transport is not the same. An example like this cannot be found in $\mathbb{L}^n$.

4.2.2 Manifolds with a finite number of Lorentz-Minkowski ends

In this section we introduce the concept of manifolds that have several Lotrentz-Minkowski ends.

Definition 4.15. *Let (M,g) be a Lorentzian manifold. We say that M has m* Lorentz-Minkowski ends *if there exists K a compact connected set such that $M \setminus \mathring{K}$ is isometric to the disjoint union of $\mathbb{L}^n \setminus \mathring{K}_i$, where K_i is a compact connected set, that is, $M \setminus \mathring{K} \cong \cup_{i=1}^m (\mathbb{L}^n \setminus \mathring{K}_i)$. If each K_i is an Euclidean ball as in definition 4.3, we say that M has m spherical Lorentz-Minkowski ends.*

We will work in the completeness of this kind of manifolds. We start studying manifolds that have spherical Lorentz-Minkowski ends, and later we generalize to regular Lorentz-Minkowski ends. First, we show an example of a manifold with two spherical Lorentz-Minkowski ends that is incomplete. In it, example 3.5 is introduced to construct a compact incomplete set, which is not trivial.

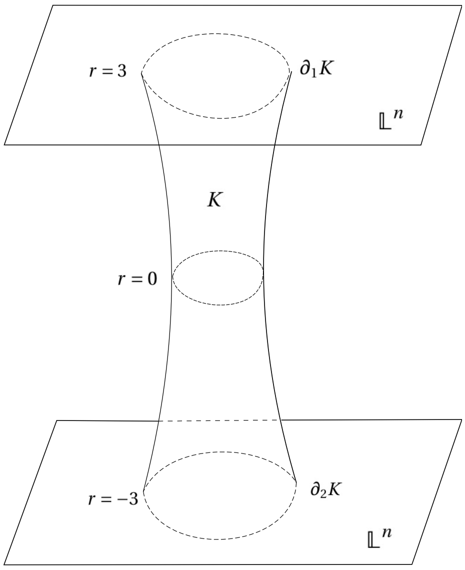


Figure 4.3: A Lorentzian manifold with two ends. In the picture, the ends are marked as $\mathbb{L}^n$.

Example 4.16. We build a Lorentzian manifold (M, g) with two ends, that is $M\backslash K = (\mathbb{L}^n \setminus B) \cup (\mathbb{L}^n \setminus B)$, where B is a compact ball. It can be represented in figure 4.3, in which the compact set is labeled as K and the Lorentz-Minkowski ends are labeled as $\mathbb{L}^n$. In this case, M is globally defined with the extended polar coordinates (r,θ) in the domain $\mathbb{R} \times [0, 2\pi]$. The ends of M are $\{r \leq -3\}$ and $\{r \geq 3\}$ and $K = \{-3 \leq r \leq 3\}$. K is a compact set homeomorphic to a cylinder, in which an incomplete metric is chosen. In the next paragraphs we build the incomplete Lorentzian metric in M. First, in the ends of M the metric is $\eta = dx^2 - dy^2$, where x, y are the usual coordinates in the Lorentz-Minkowski space. In the coordinates (r,θ), $x = r\cos\theta$ and $y = r\sin\theta$. In this coordinates, we have $dx = \cos\theta dr - r\sin\theta d\theta$ and $dy = \sin\theta dr + r\cos\theta d\theta$. Therefore, the metric in $\mathbb{L}^2$ is $\eta = \cos^2\theta dr^2 - 2r\cos\theta\sin\theta dr d\theta + r^2\sin\theta d\theta^2 - (\sin^2\theta dr^2 - 2r\cos\theta\sin\theta + r^2\cos^2\theta d\theta^2) = \cos 2\theta dr^2 - 2r\sin 2\theta dr d\theta - r^2\cos 2\theta d\theta^2$. Then, when $|r| \geq 3$, the metric of M is:

$$g = \eta = \cos 2\theta dr^2 - 2r\sin 2\theta dr d\theta - r^2\cos 2\theta d\theta^2,$$

We want to connect differentiably the metric in both ends to the following metric defined in $-1 \leq r \leq 1$:

$$g = 2dr d\theta + a(r)d\theta^2,$$

such that $a(r)$ is periodic of period 1 and $a(0) = 0$, $a'(0) \neq 0$. We are working with a metric similar to the one of example 3.5. The geodesic defined by its initial conditions $\gamma(0) = 0$, $\gamma'(0) = (0,1)$ is closed and incomplete, as in example 3.5.

To connect both metrics we will use plateau functions. We call them $h(t)$ and they are defined in the following way:

$$h(t) = \frac{u(t)}{u(t) + u(1-t)}, \quad \text{where } u(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{4.12}$$

This function is called plateau function because it is a differential function equal to zero in

$t \le 0$ and equal to 1 in $t \ge 1$. In fact, we have:

$$\lim_{t\to 0^+} h'(t) = \frac{\frac{1}{t^2}e^{-\frac{1}{t}}\left(e^{-\frac{1}{t}} - e^{-\frac{1}{1-t}}\right) + e^{-\frac{1}{t}}\left(\frac{1}{t^2}e^{-\frac{1}{t}} - \frac{1}{(1-t)^2}e^{-\frac{1}{1-t}}\right)}{\left(e^{-\frac{1}{t}} + e^{-\frac{1}{1-t}}\right)^2} = 0 = \lim_{t\to 0^-} h'(t). \tag{4.13}$$

Similarly, $h'(1) = 0$. We also need to use an important property of plateau functions. If $f(t) = g(t)h(t)$, where $g(t)$ is any differentiable function defined in $\mathbb{R}$, we have $f'(0) = 0$ and $f'(1) = g'(1)$. This is because $f'(t) = g'(t)h(t) + g(t)h'(t)$ and $h(0) = h'(0) = h'(1) = 0$, whereas $h(1) = 1$.

Using this kind of functions, the metric in $1 \le r \le 2$ can be defined as:

$$g = (-h(r-1)2r\sin 2\theta + 2h(2-r))\, dr d\theta + \left(-h(r-1)r^2\cos 2\theta + h(2-r)a(r)\right) d\theta^2.$$

This metric is clearly continuous in $r = 1$. Because of the aforementioned property of the derivatives of a plateau function we have continuity in the first derivatives as well. In $r = 2$, the metric is $2r\sin 2\theta dr d\theta - r^2\cos 2\theta d\theta^2$. The purpose now is to define the metric in $2 \le r \le 3$.

$$g = h(r-2)\cos 2\theta dr^2 - r\sin 2\theta dr d\theta - r^2\cos 2\theta d\theta^2.$$

Then, the metric is continuous in $r = 2$ and $r = 3$, and so are the first derivatives. For $-3 \le r \le -1$ a similar reasoning can be followed to complete the definition of g.

We still need to see that this metric is Lorentzian. If $|r| \le 1$ or if $|r| \ge 3$ it is clear. A sufficient condition to ensure that a metric is Lorentzian in dimension 2 is that the determinant of the metric is negative. In the case $1 \le r \le 2$, we have

$$\det g = -\left(h(r-1)2r\sin 2\theta + 2h(2-r)\right)^2 < 0.$$

Secondly, if $2 \le r \le 3$,

$$\det g = -h(r-2)r^2\cos^2 2\theta - r^2\sin^2 2\theta < 0.$$

Similarly, in $-3 \le r \le -1$, it can be obtained that $\det g < 0$. In this example we have presented a Lorentzian manifold with two spherical Lorentz-Minkowski ends that is incomplete.

If in the previous example we had chosen a complete metric in K, M would have been complete, taking for example the function $a(r)$ to be constantly zero.

We can see the completeness of this kind of manifolds by generalizing proposition 4.7.

Proposition 4.17. *Let (M, g) be a Lorentzian manifold such that M has m spherical Lorentz-Minkowski ends. If K is geodesically complete, then M is geodesically complete.*

Proof. Let $\gamma : [0, b) \to M$ be an inextendible geodesic in M. If $Im(\gamma) \subset M \setminus \mathring{K}$, as γ is connected, $Im(\gamma) \subset \mathbb{L}^n \setminus \mathring{K}_i$. In this case, γ is an straight line and thus, complete. In case $Im(\gamma) \cap K \ne \emptyset$, there are two possibilities. If γ enters K and it does not leave again, γ is complete by geodesic completeness of K. Otherwise, γ leaves at least once. In this case, it leaves to one of the ends, and from the convexity of ∂K_i follows that γ does not return to it. $\square$

The theorem 4.8 can also be generalized to this type of manifolds.

Theorem 4.18. *Under the hypotheses of proposition 4.17, there exists $U_K(g)$ a neighborhood of Lorentzian metrics up to K such that if $\bar{g} \in U_K(g)$, $(M, \bar{g})$ is complete.*

Proof. The proof of this theorem is based on the proof of theorem 4.8. In each end $\mathbb{L}^n \setminus B_i$, two neighborhoods of metrics can be taken, fulfilling the same conditions as $U_1(g)$ and $U_2(g)$ in the proof of theorem 4.8. For each end $\mathbb{L}^n \setminus B_i$ we have the neighborhoods:

- $U_1^i(g) = \{\bar{g} \in \mathrm{Lor}(\mathbb{L}^n \setminus \mathring{K}_i) : |g - \bar{g}|_1 < \delta_1^i\}$ in which $\mathbb{L}^n \setminus K_i$ is complete.
- $U_2^i(g) = \{\bar{g} \in \mathrm{Lor}(\mathbb{L}^n \setminus \mathring{K}_i) : |g - \bar{g}|_1 < \delta_2^i\}$ in which affine spheres are strongly convex in $\mathbb{L}^n \setminus B_i$.

Having these neighborhoods, we can build a neighborhood of metrics up to K of g. First, the function:

$$\delta(p) = \begin{cases} 0 & p \in K \\ e^{\frac{1}{R(p)-R_i}} \min(\delta_1^i, \delta_2^i) & p \in \mathbb{L}^n \setminus K_i \quad i = 1, \ldots, m. \end{cases}$$

is defined in M and is continuous and zero in K. In the definition of this function we have used t_i that is the radius of K_i and the function t, defined in each end, is the radius function in $\mathbb{L}^n$. Then,

$$U_K(g) = \{\bar{g} \in \mathrm{Lor}(M) : |\bar{g} - g|_1 \leq \delta \text{ in } K, |\bar{g} - g|_1 < \delta \text{ in } M \setminus K\} \tag{4.14}$$

is a neighborhood of metrics up to K. Completeness in this neighborhood has to be checked. Ler $\gamma : [0, b) \to M$ be a geodesic in $(M, \bar{g})$ with $\bar{g} \in U_K(g)$.

- If γ is imprisoned in K, γ is complete because $\bar{g} = g$ in K.
- If $Im(\gamma) \cap K \neq \emptyset$ and γ points outwards at some point $\gamma(s_0)$ in ∂K_i for some $i \in \{1, \ldots, m\}$, we have that from that point the geodesic is a geodesic leaving K_i, which is a convex set. Furthermore, the collection of spheres $\{S_R\}_R$ are convex sets in $\mathbb{L}^n \setminus K_i$ because $\bar{g} \in U_2^i(g)$, γ does not return to K. As $\bar{g} \in U_1^i(g)$, $\mathbb{L}^n \setminus K_i$ is geodesically complete, γ is complete.
- If $Im(\gamma) \in \mathbb{L}^n \setminus K_i$ for some i, γ is complete because $\bar{g} \in U_1^i(g)$ in $\mathbb{L}^n \setminus K_i$.

□

In the line of section 4.2.1, each $\partial_i K$ does not have to be an sphere and can be any connected hypersurface in the i-th copy of $\mathbb{L}^n$. In the remainder of the section, we work with manifolds with m Lorentz-Minkowski ends. In the special case that K has precompact holonomy we can ensure its completeness and the stability of its completeness out of a compact set $\bar{K}$ with the characteristics of the one in definition 4.15 that contains K.

Proposition 4.19. *Let (M, g) be a Lorentzian manifold with m Lorentz-Minkowski ends. If $M \setminus \mathring{K} = \cup_{i=1}^m (\mathbb{L}^n \setminus \mathring{K}_i)$ and K has precompact holonomy, (M, g) is complete.*

Proof. We do a recursive application of lemma 4.13. Let $\mathbb{L}^n \setminus \mathring{K}_1$ be an end of M. Defining the connected manifold $\mathbb{L}^n \setminus \mathring{K}_1 \cup K$ we are in the conditions of lemma 4.13, therefore, there exists a compact ball B_1 that encloses $\partial_1 K$ and has precompact holonomy. We define $\bar{K}_1 =$

$(B_1 \setminus \mathring{K}_1) \cup K$ and the Lorentz-Minkowski end $\mathbb{L}^n \setminus B_1$. We proceed in the same way for the rest of the ends, using $\bar{K}_i$ as the compact set for lemma 4.13, until we find $M \setminus \bar{K}_m = \cup_{i=1}^m (\mathbb{L}^n \setminus \mathring{B}_i)$. Hence, M has m Lorentz-Minkowski ends. Besides, by construction, $\bar{K}_m$ has precompact holonomy. Using proposition 4.17, we have completeness of M. □

Theorem 4.20. *Let (M, g) be a Lorentzian manifold with m Lorentz-Minkowski ends. If $M \setminus \mathring{K} = \cup_{i=1}^m (\mathbb{L}^n \setminus \mathring{K}_i)$ and K has precompact holonomy, there exists a compact set $\bar{K}$ that contains K, such that there exists $U_{\bar{K}}(g)$ a neighborhood of metrics up to $\bar{K}$ such that $(M, \bar{g})$ is complete for $\bar{g} \in U_{\bar{K}}(g)$.*

Proof. By proposition 4.19, M is complete and taking $\bar{K} = \bar{K}_m$ in the proof of this proposition, we have $M \setminus \bar{K} = \cup_{i=1}^m (\mathbb{L}^n \setminus \mathring{B}_i)$, therefore, M has m spherical Lorentz-Minkowski ends. Applying theorem 4.18, we have a neighborhood of metrics up to $\bar{K}$ such that if $\bar{g} \in U_{\bar{K}}(g)$, $(M, \bar{g})$ is complete. □

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