DUALITY PROBLEMS IN LIPSCHITZ FUNCTION SPACES

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Abstract. We address some problems from the duality theory in spaces of Lipschitz functions $\mathrm{Lip}_0(X)$. The preduality problem of $\mathrm{Lip}_0(X)$ is analysed by applying the Dixmier–Ng theorem [2], and this approach permits us to describe the closed unit ball of the Lipschitz-free space of X as the closed convex balanced hull in $\mathrm{Lip}_0(X)^*$ of the Lipschitz evaluation functionals. Inspired by a theorem of Weaver [4], we define and study the concept of a W-linearization for $\mathrm{Lip}_0(X)$ which is a linearizing construction stronger than a predual space. Building upon the ideas of Aron and Schottenloher [1] and Ryan [3], we introduce the notions of a compact Lipschitz map and a weakly compact Lipschitz map from a metric space X into a Banach space E. A theory of such maps which closely parallels the theory for linear operators is developed and the approximation property of $\mathrm{Lip}_0(X)$ is characterized. In terms of the transpose map of a Lipschitz map, we state Lipschitz versions of Schauder type theorems on the (weak) compactness of the adjoint of a (weakly) compact linear operator. Furthermore, it is proved that the map from $\mathrm{Lip}_0(X,E)$ to $\mathcal{L}(E^*,\mathrm{Lip}_0(X))$ that sends each function to its transpose is an isometric isomorphism from $\mathrm{Lip}_0(X,E)$ onto $\mathcal{L}((E^*,w^*);(\mathrm{Lip}_0(X),w^*))$, as well as from the space of compact Lipschitz maps (weakly compact Lipschitz maps) from X to E onto the space $\mathcal{L}((E^*,bw^*);\mathrm{Lip}_0(X))$ (respectively, $\mathcal{L}((E^*,w^*);(\mathrm{Lip}_0(X),w))$). When X is compact, the same map identifies the little Lipschitz space $\mathrm{lip}_0(X,E)$ with $\mathcal{L}((E^*,bw^*);\mathrm{lip}_0(X))$. For compact metric spaces X, we tackle the biduality problem as to when $\mathrm{Lip}_0(X)$ is isometrically isomorphic to $\mathrm{lip}_0(X)^{**}$, and show that this is the case whenever the closed unit ball of $\mathrm{lip}_0(X)$ is dense in the closed unit ball of $\mathrm{Lip}_0(X)$ with respect to the topology of pointwise convergence.

References

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